Dot Product

CSE 457, Autumn 2003
Graphics

http://www.cs.washington.edu/education/courses/457/03au/
Readings and References

Readings

- Sections 1.3.4, 1.3.5, 3D Computer Graphics, Watt

Other References

- Section A.3, Dot Products and Distances, Computer Graphics, Principles and Practice, Foley, van Dam
Dot Product

• The dot product or inner product of two vectors is a very useful operation in computer graphics and is applied in numerous ways

• These notes are a short review of what it the dot product is and some examples of how it gets used
Definition

\[ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \]

\[ v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3 = \|v\| \|w\| \cos(\theta) \]

If \( v \) is a unit vector, then

\[ v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3 = \|w\| \cos(\theta) \]

and so \( v \cdot w \) is the length of the projection of \( w \) onto \( v \)
Illustration of $V \cdot W$

\[ V \cdot W = V_x W_x + V_y W_y \]
\[ = \|V\| \cos(\alpha_v) \|W\| \cos(\alpha_w) + \|V\| \sin(\alpha_v) \|W\| \sin(\alpha_w) \]
\[ = \|V\| \|W\| \left[ \cos(\alpha_v) \cos(\alpha_w) + \sin(\alpha_v) \sin(\alpha_w) \right] \]
\[ = \|V\| \|W\| \cos(\alpha_w - \alpha_v) \]
\[ = \|V\| \|W\| \cos(\theta) \]

\[ V_x = \|V\| \cos(\alpha_v) \]
\[ V_y = \|V\| \sin(\alpha_v) \]
\[ W_x = \|W\| \cos(\alpha_w) \]
\[ W_y = \|W\| \sin(\alpha_w) \]
The cosine is a useful function ...

if both \( v \) and \( w \) are unit vectors, then

\[
v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3
= \|v\|\|w\| \cos(\theta)
= \cos(\theta)
\]

and so \( v \cdot w \) is just the cosine of the angle between the vectors

\[\theta < 90^\circ, \cos(\theta) > 0\]

\[\theta > 90^\circ, \cos(\theta) < 0\]

\[\theta = 90^\circ, \cos(\theta) = 0\]
Unit vectors

The dot product of $\mathbf{v}$ with itself is

$$\mathbf{v} \cdot \mathbf{v} = v_1v_1 + v_2v_2 + v_3v_3$$

$$= \|\mathbf{v}\|^2 \cos(0)$$

$$= \|\mathbf{v}\|^2$$

and so $\mathbf{v} \cdot \mathbf{v}$ is the square of its length

and if $\mathbf{v}$ is a unit vector then $\mathbf{v} \cdot \mathbf{v}$ is 1

the columns of a rotation matrix are perpendicular unit vectors

$$\mathbf{a} \cdot \mathbf{a} = \cos \theta \cos \theta + \sin \theta \sin \theta = 1$$

$$\mathbf{a} \cdot \mathbf{b} = \cos \theta (-\sin \theta) + \sin \theta \cos \theta = 0$$

and so the transpose of a rotation matrix

is its inverse

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

notice that the transpose is also rotation through $-\theta$, since $-\sin(-\theta) = \sin(\theta)$
Front facing polygon?

surface normal dot product with ray direction
Equation of a line

$$Ax + By + C = 0$$

All vectors \((x,y)\) for which \((A,B) \cdot (x,y) = -C\)
Where on ray is closest approach to C?

\[ \text{RC} = C - R_o \]

*RC* is ray direction vector (unit vector)

so

\[ t_{ca} = \text{RC} \cdot R_{dir} \]
Diffuse reflection of light

\[ A_p = A \cos \theta \]

\[ \frac{A_p}{A} = \cos \theta \]

\[ N \cdot L = \|L\| \cos \theta \]

so \( N \cdot L \) gives a scaled value for diffuse reflected light intensity