Geometric transformations

Geometric transformations will map points in one space to points in another: \((x', y', z') = f(x, y, z)\).

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We’ll focus on transformations that can be represented easily with matrix operations.

We’ll start in 2D...

Representation

We can represent a point, \(p = (x, y)\), in the plane

- as a column vector
  
  \[
  \begin{bmatrix}
  x \\
  y
  \end{bmatrix}
  \]

- as a row vector
  
  \[
  [x \ y]
  \]
Representation, cont.

We can represent a 2-D transformation $M$ by a matrix

$$
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
$$

If $p$ is a column vector, $M$ goes on the left:

$$
p' = Mp
$$

$$
\begin{bmatrix}
    x' \\
    y'
\end{bmatrix} =
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
\begin{bmatrix}
    x \\
    y
\end{bmatrix}
$$

If $p$ is a row vector, $M^T$ goes on the right:

$$
p' = pM^T
$$

$$
\begin{bmatrix}
    x' & y'
\end{bmatrix} =
\begin{bmatrix}
    x & y
\end{bmatrix}
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
$$

We will use column vectors.

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix $M$:

$$
\begin{bmatrix}
    x' \\
    y'
\end{bmatrix} =
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
\begin{bmatrix}
    x \\
    y
\end{bmatrix}
$$

So:

$$
x' = ax + by
$$

$$
y' = cx + dy
$$

We will develop some intimacy with the elements $a, b, c, d…$

Identity

Suppose we choose $a=d=1, b=c=0$:

- Gives the identity matrix:

$$
\begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix}
$$

- Doesn’t move the points at all

Scaling

Suppose we set $b=c=0$, but let $a$ and $d$ take on any positive value:

- Gives a scaling matrix:

$$
\begin{bmatrix}
    a & 0 \\
    0 & d
\end{bmatrix}
$$

- Provides differential scaling in $x$ and $y$:

$$
x' = ax
$$

$$
y' = dy
$$
Suppose we keep $b=c=0$, but let either $a$ or $d$ go negative.

Examples:

\[
\begin{bmatrix}
-1 & 0 \\
0 & 1 \\
\end{bmatrix}
\quad \quad \quad
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}
\]

Now let’s leave $a=d=1$ and experiment $b\ldots$

The matrix

\[
\begin{bmatrix}
1 & b \\
0 & 1 \\
\end{bmatrix}
\]

gives:

\[
x' = x + by \\
y' = y
\]

Effect on unit square

Let’s see how a general 2 x 2 transformation $M$ affects the unit square:

\[
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix}
\begin{bmatrix}
p & q & r & s \\
\end{bmatrix} =
\begin{bmatrix}
p' & q' & r' & s' \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix} =
\begin{bmatrix}
0 & a & a+b & b \\
0 & c & c+d & d \\
\end{bmatrix}
\]

Observe:

- Origin invariant under $M$
- $M$ can be determined just by knowing how the corners (1,0) and (0,1) are mapped
- $a$ and $d$ give $x$- and $y$-scaling
- $b$ and $c$ give $x$- and $y$-shearing
Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for “rotation about the origin”:

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

Thus,

\[
M = R(\theta) = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

Limitations of the 2 x 2 matrix

A 2 x 2 matrix allows
- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

Homogeneous coordinates

Idea is to loft the problem up into 3-space, adding a third component to every point:

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} \rightarrow \begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]

And then transform with a 3 x 3 matrix:

\[
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix} = T(t) \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
1 & 0 & t_x \\
0 & 1 & t_y \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]

… gives translation!

Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, $\theta$, about any point $q = [q_x \ q_y]^T$ with a matrix:

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & \frac{1}{2} \\
0 & 0 & 1
\end{pmatrix}
\]

1. Translate $q$ to origin
2. Rotate
3. Translate back

Note: Transformation order is important!!
Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

For example, scaling:

\[ \frac{\mathbf{x}^{'}}{w'} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \]

Translation in 3D

\[ \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \]

Rotation in 3D

Rotation now has more possibilities in 3D:

\[ R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

Shearing in 3D

Shearing is also more complicated. Here is one example:

\[ \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \]

Use right hand rule
Properties of affine transformations

All of the transformations we've looked at so far are examples of "affine transformations."

Here are some useful properties of affine transformations:
- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)

\[
\text{ratio} = \frac{\|pq\|}{\|qr\|} = \frac{s}{t} = \frac{\|p'q'\|}{\|q'r'\|}
\]

Summary

What to take away from this lecture:
- All the names in boldface.
- How points and transformations are represented.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.