Review of Eigenvectors and Eigenvalues

from CliffsNotes Online
http://www.cliffsnotes.com/study_guide/Determining-the-Eigenvectors-of-a-Matrix.topicArticleId-20807,articleId-20804.html
Definition

The eigenvectors $\mathbf{x}$ and eigenvalues $\lambda$ of a matrix $A$ satisfy

$$A\mathbf{x} = \lambda \mathbf{x}$$

If $A$ is an $n \times n$ matrix, then $\mathbf{x}$ is an $n \times 1$ vector, and $\lambda$ is a constant.

The equation can be rewritten as $(A - \lambda I) \mathbf{x} = 0$, where $I$ is the $n \times n$ identity matrix.
Computing Eigenvalues

Since $\mathbf{x}$ is required to be nonzero, the eigenvalues must satisfy

$$\det(A - \lambda I) = 0$$

which is called the *characteristic equation*. Solving it for values of $\lambda$ gives the eigenvalues of matrix $A$. 
2 X 2 Example

\[
A = \begin{bmatrix}
1 & -2 \\
3 & -4
\end{bmatrix} \quad \text{so} \quad A - \lambda I = \begin{bmatrix}
1 - \lambda & -2 \\
3 & -4 - \lambda
\end{bmatrix}
\]

\[
det(A - \lambda I) = (1 - \lambda)(-4 - \lambda) - (3)(-2) \\
= \lambda^2 + 3 \lambda + 2
\]

Set $\lambda^2 + 3 \lambda + 2$ to 0

Then $\lambda = (-3 \pm/\sqrt{9-8})/2$

So the two values of $\lambda$ are -1 and -2.
Finding the Eigenvectors

Once you have the eigenvalues, you can plug them into the equation $Ax = \lambda x$ to find the corresponding sets of eigenvectors $x$.

\[
\begin{bmatrix}
1 & -2 \\
3 & -4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
=
-1
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

so

\[
\begin{align*}
x_1 - 2x_2 &= -x_1 \\
3x_1 - 4x_2 &= -x_2
\end{align*}
\]

These equations are not independent. If you multiply (2) by $2/3$, you get (1).

The simplest form of (1) and (2) is $x_1 - x_2 = 0$, or just $x_1 = x_2$. 

Since $x_1 = x_2$, we can represent all eigenvectors for eigenvalue -1 as multiples of a simple basis vector:

$$E = t \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $t$ is a parameter.


For the second eigenvalue (-2) we get

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so

$$x_1 - 2x_2 = -2x_1$$
$$3x_1 - 4x_2 = -2x_2$$

(1) $3x_1 - 2x_2 = 0$
(2) $3x_1 - 2x_2 = 0$

so eigenvectors are of the form $t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. 
Generalization for 2 X 2 Matrices

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\lambda = (a + d) \pm \sqrt{(a+d)^2 - 4(ad - bc)} \over 2$

The discriminant (the part under the square root), can be simplified to get $\sqrt{(a-d)^2 + 4bc}$.

If $b = c$, this becomes $\sqrt{(a-d)^2 + (2b)^2}$
Since the discriminant is the sum of 2 squares, it has real roots.

We will be seeing some 2 x 2 matrices where indeed $b = c$, so we’ll be guaranteed a real-valued solution for the eigenvalues.
Another observation we will use:

For 2 x 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$\lambda_1 + \lambda_2 = a + d$, which is called $\text{trace}(A)$
and
$\lambda_1\lambda_2 = ad - bc$, which is called $\text{det}(A)$.

Finally, zero is an eigenvalue of $A$ if and only if $A$ is singular and $\text{det}(A) = 0$. 