Markov Decision Processes (MDPs)

Machine Learning – CSE446
Carlos Guestrin
University of Washington

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Reinforcement Learning

training by feedback

Learning to act

- Reinforcement learning
- An agent
  - Makes sensor observations
  - Must select action
  - Receives rewards
    - positive for "good" states
    - negative for "bad" states

[Ng et al. '05]

Markov Decision Process (MDP) Representation

- State space:
  - Joint state **x** of entire system
- Action space:
  - Joint action **a**= $\{a_1,\ldots, a_n\}$ for all agents
- Reward function:
  - Total reward R(**x**,**a**)
    - sometimes reward can depend on action
- Transition model:
  - Dynamics of the entire system P(**x**'|**x**,**a**)

Discount Factors

People in economics and probabilistic decision-making do this all the time.

The "Discounted sum of future rewards" using discount factor γ" is

(reward now) +

γ (reward in 1 time step) +

γ^2 (reward in 2 time steps) +

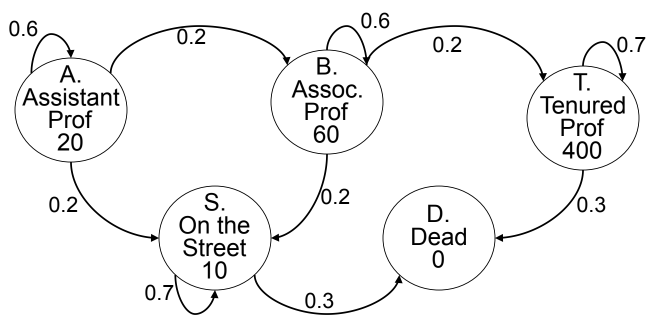
γ^3 (reward in 3 time steps) +

:

: (infinite sum)

The Academic Life

Assume Discount Factor $\gamma = 0.9$



Define:

V_A = Expected discounted future rewards starting in state A

V_B = Expected discounted future rewards starting in state B

V_T = " " " " " " " T

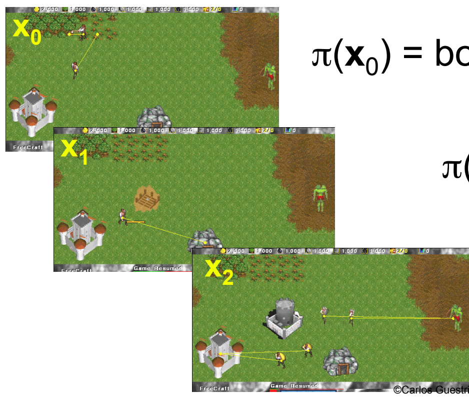
V_S = " " " " " " " S

V_D = " " " " " " " D

How do we compute V_A, V_B, V_T, V_S, V_D ?

Policy

Policy: $\pi(\mathbf{x}) = \mathbf{a}$ → At state $\mathbf{x}$, action $\mathbf{a}$ for all agents



$\pi(\mathbf{x}_0)$ = both peasants get wood

$\pi(\mathbf{x}_1)$ = one peasant builds barrack, other gets gold

$\pi(\mathbf{x}_2)$ = peasants get gold, footmen attack

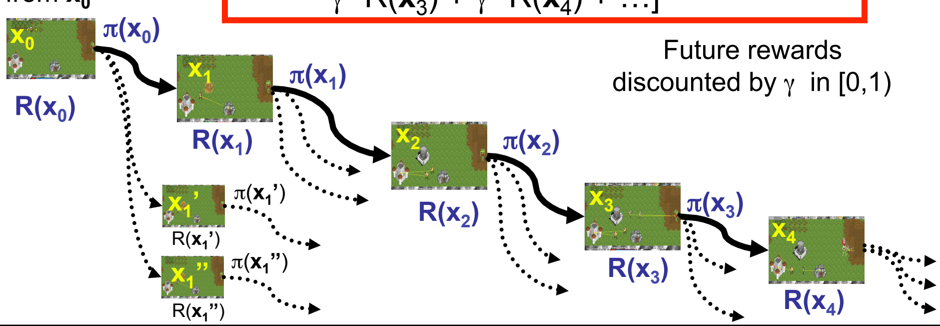
Value of Policy

Value: $V_\pi(\mathbf{x})$ → Expected long-term reward starting from $\mathbf{x}$

$$V_\pi(\mathbf{x_0}) = \mathbf{E}_\pi[R(\mathbf{x}_0) + \gamma\, R(\mathbf{x}_1) + \gamma^2\, R(\mathbf{x}_2) + \gamma^3\, R(\mathbf{x}_3) + \gamma^4\, R(\mathbf{x}_4) + \ldots]$$

Start from $\mathbf{x_0}$

Future rewards discounted by γ in [0,1)



Computing the value of a policy

$V_\pi(\mathbf{x_0}) = \mathbf{E}_\pi[R(\mathbf{x}_0) + \gamma\ R(\mathbf{x}_1) + \gamma^2\ R(\mathbf{x}_2) + \gamma^3\ R(\mathbf{x}_3) + \gamma^4\ R(\mathbf{x}_4) + \ldots]$

- Discounted value of a state:
  - value of starting from x_0 and continuing with policy π from then on

$$\begin{aligned} V_\pi(x_0) &= E_\pi[R(x_0) + \gamma R(x_1) + \gamma^2 R(x_2) + \gamma^3 R(x_3) + \cdots] \\ &= E_\pi[\sum_{t=0}^{\infty} \gamma^t R(x_t)] \end{aligned}$$

- A recursion!

Simple approach for computing the value of a policy: Iteratively

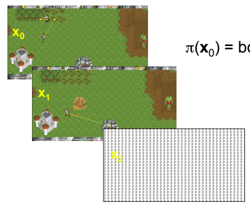
$$V_\pi(x) = R(x) + \gamma \sum_{x'} P(x' \mid x, a = \pi(x)) V_\pi(x')$$

- Can solve using a simple convergent iterative approach: (a.k.a. dynamic programming)
  - Start with some guess V^0
  - Iteratively say:
    - $V_\pi^{t+1}(x) \leftarrow R(x) + \gamma \sum_{x'} P(x' \mid x, a = \pi(x)) V_\pi^t(x')$
  - Stop when $||V_{t+1}-V_t||_\infty < \varepsilon$
    - means that $||V_\pi-V_{t+1}||_\infty < \varepsilon/(1-\gamma)$

But we want to learn a **Policy**

- So far, told you how good a policy is…
- But how can we choose the best policy???
- Suppose there was only one time step:
  - world is about to end!!!
  - select action that maximizes reward!

Policy: $\pi(\mathbf{x}) = \mathbf{a}$ → At state **x**, action **a** for all agents



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Unrolling the recursion

- Choose actions that lead to best value in the long run
  - Optimal value policy achieves optimal value V^*

$$V^*(x_0) \;=\; \max_{a_0} R(x_0,a_0) + \gamma E_{a_0}[\max_{a_1} R(x_1) + \gamma^2 E_{a_1}[\max_{a_2} R(x_2) + \cdots]]$$

Bellman equation

- Evaluating policy π:

$$V_\pi(x) \quad = \quad R(x) + \gamma \sum_{x'} P(x' \mid x, a = \pi(x)) V_\pi(x')$$

- Computing the optimal value V^* - Bellman equation

$$V^*(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^*(\mathbf{x}')$$

Optimal Long-term Plan

Optimal value function $V^*(\mathbf{x})$

Optimal Policy: $\pi^*(\mathbf{x})$

Optimal policy:

$$\pi^*(\mathbf{x}) = \arg\max_{a} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^*(\mathbf{x}')$$

Interesting fact – Unique value

$$V^*(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^*(\mathbf{x}')$$

- *Slightly surprising fact*: There is only one V^* that solves Bellman equation!
  - there may be many optimal policies that achieve V^*
- *Surprising fact*: optimal policies are good everywhere!!!

$$V_{\pi^*}(x) \geq V_\pi(x), \quad \forall x, \quad \forall \pi$$

Solving an MDP

Solve Bellman equation

Optimal value $V^*(\mathbf{x})$

Optimal policy $\pi^*(\mathbf{x})$

$$V^*(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^*(\mathbf{x}')$$

Bellman equation is non-linear!!!

Many algorithms solve the Bellman equations:

- Policy iteration [Howard '60, Bellman '57]
- Value iteration [Bellman '57]
- Linear programming [Manne '60]
- …

Value iteration (a.k.a. dynamic programming) – the simplest of all

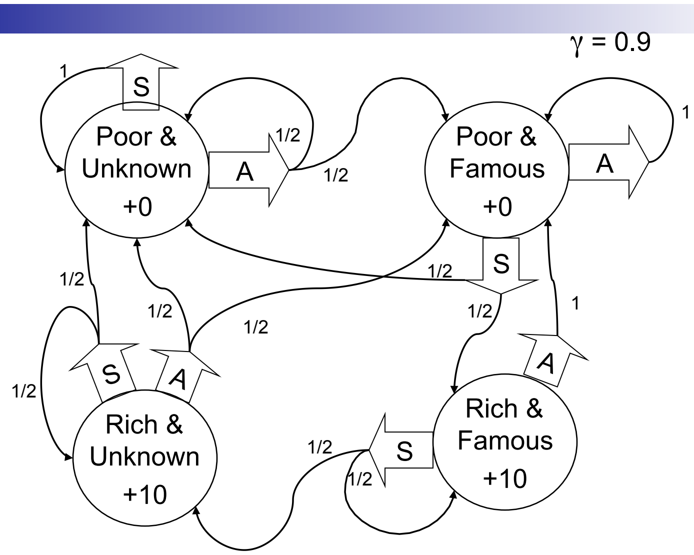
$$V^*(x) \;=\; R(x,a) + \gamma \sum_{x'} P(x' \mid x, a = \pi(x)) V^*(x')$$

- Start with some guess V^0
- Iteratively say:
  - $V^{t+1}(x) \;\leftarrow\; \max_a R(x,a) + \gamma \sum_{x'} P(x' \mid x, a) V^t(x')$
- Stop when $||V_{t+1}-V_t||_\infty < \varepsilon$
  - means that $||V^*-V_{t+1}||_\infty < \varepsilon/(1-\gamma)$

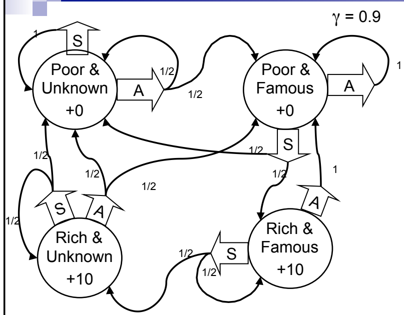
A simple example

You run a startup company.

In every state you must choose between Saving money or Advertising.



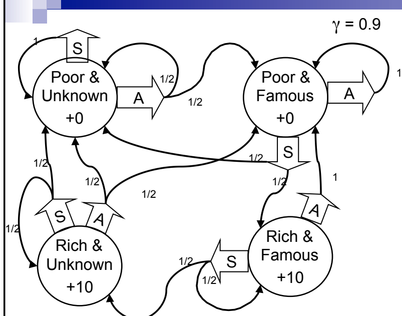
Let's compute $V_t(x)$ for our example



t	V^t(PU)	V^t(PF)	V^t(RU)	V^t(RF)
1				
2				
3				
4				
5				
6				

$$V^{t+1}(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^t(\mathbf{x}')$$

Let's compute $V_t(x)$ for our example



t	V^t(PU)	V^t(PF)	V^t(RF)	V^t(RU)
1	0	0	10	10
2	0	4.5	14.5	19
3	2.03	6.53	25.08	18.55
4	3.852	12.20	29.63	19.26
5	7.22	15.07	32.00	20.40
6	10.03	17.65	33.58	22.43

$$V^{t+1}(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^t(\mathbf{x}')$$

What you need to know

- What's a Markov decision process
  - state, actions, transitions, rewards
  - a policy
  - value function for a policy
    - computing V_π
- Optimal value function and optimal policy
  - Bellman equation
- Solving Bellman equation
  - with value iteration, policy iteration and linear programming

Acknowledgment

- This lecture contains some material from Andrew Moore's excellent collection of ML tutorials:
  - http://www.cs.cmu.edu/~awm/tutorials