Reading assignment: Read section 7.5 and sections 8.1-8.2.

Problems:

1. Show that if $P = NP$ then every language $A \in P$, except $A = \emptyset$ and $A = \Sigma^*$, is NP-complete.

2. Let $U = \{ \langle M, x, 1^t \rangle \mid M$ is an NTM that accepts input $x$ within $t$ steps$\}$. Show that $U$ is $NP$-complete.

3. Let $\phi$ be a 3CNF-formula. An NOTALLEQUAL assignment to the variables of $\phi$ is one where each clause contains two literals with different truth values. In other words a NOTALLEQUAL assignment satisfies $\phi$ but does not set all three literals to true in any clause.

   (a) Show that the negation of a NOTALLEQUAL assignment for $\phi$ is also a NOTALLEQUAL assignment for $\phi$.

   (b) Let $NAESAT$ be the set of all 3CNF formulas $\phi$ that have a NOTALLEQUAL assignment. Prove that $NAESAT$ is NP-complete. For the hardness part use a reduction from 3SAT.

   (Hint: Show that the function that replaces each clause $C_i$ of $\phi$ of the form $(y_1 \lor y_2 \lor y_3)$ where $y_1, y_2, y_3$ are literals by the two clauses $(y_1 \lor y_2 \lor z_i)$ and $(z_i \lor y_3 \lor w)$ where $w$ is a single new variable for all clauses and there is one $z_i$ variable per original clause.)

4. A cut in an undirected graph $G$ is a partition of the vertices $V$ of $G$ into two disjoint parts $S$ and $T$ with $V = S \cup T$. The size of the cut $(S, T)$ is the number of edges that cross between $S$ and $T$. Define

   $\text{MAXCUT} = \{ \langle G, k \rangle \mid G$ has a cut of size $\geq k$\}.

Show that $\text{MAXCUT}$ is NP-complete. For the hardness part use the fact that $NAESAT$ is NP-complete.

(Hint: For each variable $x$ in an m-clause 3CNF formula, have $3m$ vertices for $x$ and $3m$ vertices for $\overline{x}$ for a total of $6m$ vertices in the graph. Join each pair of nodes with the same variable label, but different signs by an edge. For each clause, consider a separate vertex of every possible literal label to be dedicated to that clause. Among those literals, add edges that join the 3 literals that actually appear in the clause to form a triangle. What value of $k$ should you use? Prove that this works.)
5. Let $01\text{ROOT} = \{ \langle p \rangle \mid p$ is a polynomial in $n$ variables with integer coefficients such that $p(x_1, \ldots, x_n) = 0$ for some assignment $(x_1, \ldots, x_n) \in \{0, 1\}^n \}$. 
   
   (a) Show that $01\text{ROOT} \in \text{NP}$.  
   
   (b) Show that $3\text{SAT} \leq_P 01\text{ROOT}$. (HINT: First figure out how to convert each clause into a polynomial that evaluates to 0 iff the clause is satisfied. Then create a polynomial $q$ that evaluates to 0 if and only if all of its inputs are 0. Finally, figure out how to combine the individual polynomials for the clauses using the polynomial $q$.

6. (Extra credit) Recall that a 2-CNF formula is a CNF formula in which each clause has 2 literals and that $2\text{-SAT} = \{ \langle \phi \rangle \mid \phi$ is a satisfiable 2-CNF formula\}. Show that $2\text{SAT} \in P$. 