Graph Coloring

Input: An undirected graph \( G = (V, E) \) and a number \( K \)
Output: Decide whether \( G \) has a \( K \)-coloring (No edge between same color nodes)

Example:

3 COL: \{ <G>: G has a proper 3-Coloring \}

Planar Graph: Graph that can be drawn in the Euclidean Plane without edge crossings

Facts

- 1-Coloring \( \iff \) Graph has no edge

- 2-Coloring \( \iff \) Graph is bipartite (Graph has no odd cycle)
  Example:

- 4-Coloring: Every planar graph!
**THM: 3 COL is NP-Complete**

Proof:
1. 3 COL ∈ NP
   (Easy! 😊)

2. 3 SAT ≤ₚ 3 COL

Goal:
Convert Ø => (Poly-time) G, such that
Ø is satisfiable ⇔ G has a 3-Coloring

Ø = C₁ ^ C₂ ^ … ^ Cₘ
Eg: Where Cᵢ = X₇ ∨ X₉ ∨ X₅
n variables: X₁, X₂, … , Xₙ

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![Diagram](attachment:diagram.png)
Inset Or-Gadget:

Example:

Assume $\emptyset$ has a satisfy assignment
$X_2 = F, \ X_7 = T, \ X_9 = F$
Then $G$ has 3 coloring as follow:
\[ 3 \text{COL} \leq_p \text{PLANAR-3 COL} \]

\text{PLANAR-3COL} = \{ \langle G \rangle : \text{G is planar and has a 3-Coloring} \} \\

Goal: Put a gadget in every across, make \( u \) and \( v \) have different color  
\hspace{1cm} \text{x and y have different color} \\

\hspace{1cm} \begin{array}{c}
  \text{x} \\
  \text{y} \\
  \text{v} \\
\end{array} \\
\hspace{1cm} \Rightarrow \\

\hspace{1cm} \begin{array}{c}
  u \\
\end{array} \\

\text{TBC…}