CSE 431:

More NP-completeness

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We already know

- $3SAT \leq_p CLIQUE$
- CIRCUIT-SAT is NP-complete
- We now show Cook-Levin Theorem that $3SAT$ is NP-complete
A useful property of polynomial-time reductions

- **Theorem:** If $A \leq_p B$ and $B \leq_p C$ then $A \leq_p C$

- **Proof idea:**
  - Compose the reduction $f$ from $A$ to $B$ with the reduction $g$ from $B$ to $C$ to get a new reduction $h(x) = g(f(x))$ from $A$ to $C$.
  - The general case is similar and uses the fact that the composition of two polynomials is also a polynomial.
Theorem (Cook 1971, Levin 1973): 3-SAT is \textbf{NP}-complete

Corollary: B is \textbf{NP}-hard \iff 3-SAT \leq_p B
  (or A \leq_p B for any \textbf{NP}-complete problem A)

Proof:
  - If B is \textbf{NP}-hard then every problem in \textbf{NP}
polynomial-time reduces to B, in particular 3-SAT
does since it is in \textbf{NP}
  - For any problem A in \textbf{NP}, A \leq_p 3-SAT and so if
3-SAT \leq_p B we have A \leq_p B.
    - therefore B is \textbf{NP}-hard if 3-SAT \leq_p B
Reductions by Simple Equivalence

Show: Clique \( \leq_p \) Independent-Set

Clique:
- Given a graph \( G=(V,E) \) and an integer \( k \), is there a subset \( U \) of \( V \) with \( |U| \geq k \) such that every pair of vertices in \( U \) is joined by an edge?

Independent-Set:
- Given a graph \( G=(V,E) \) and an integer \( k \), is there a subset \( U \) of \( V \) with \( |U| \geq k \) such that no two vertices in \( U \) are joined by an edge?
Clique $\leq_p$ Independent-Set

- Given $(G, k)$ as input to Independent-Set where $G = (V, E)$
- Transform to $(G', k)$ where $G' = (V, E')$ has the same vertices as $G$ but $E'$ consists of precisely those edges that are not edges of $G$
- $U$ is an independent set in $G$
- $U$ is a clique in $G'$
More Reductions

Show: Independent Set \( \leq_p \) Vertex-Cover

Vertex-Cover:

- Given an undirected graph \( G=(V,E) \) and an integer \( k \) is there a subset \( W \) of \( V \) of size at most \( k \) such that every edge of \( G \) has at least one endpoint in \( W \)? (i.e. \( W \) covers all edges of \( G \))? 

Independent-Set:

- Given a graph \( G=(V,E) \) and an integer \( k \), is there a subset \( U \) of \( V \) with \( |U| \geq k \) such that no two vertices in \( U \) are joined by an edge?
Claim: In a graph \( G=(V,E) \), \( S \) is an independent set iff \( V-S \) is a vertex cover.

Proof:

\( \Rightarrow \) Let \( S \) be an independent set in \( G \).
- Then \( S \) contains at most one endpoint of each edge of \( G \).
- At least one endpoint must be in \( V-S \).
- \( V-S \) is a vertex cover.

\( \Leftarrow \) Let \( W=V-S \) be a vertex cover of \( G \).
- Then \( S \) does not contain both endpoints of any edge (else \( W \) would miss that edge).
- \( S \) is an independent set.
Reduction

- Map $(G, k)$ to $(G, n-k)$
  - Previous lemma proves correctness

- Clearly polynomial time

- We also get that
  - $\text{Vertex-Cover} \leq_p \text{Independent Set}$
Show: Vertex-Cover $\leq_p$ Set-Cover

Vertex-Cover:
- Given an undirected graph $G = (V, E)$ and an integer $k$ is there a subset $W$ of $V$ of size at most $k$ such that every edge of $G$ has at least one endpoint in $W$? (i.e. $W$ covers all edges of $G$)?

Set-Cover:
- Given a set $U$ of $n$ elements, a collection $S_1, \ldots, S_m$ of subsets of $U$, and an integer $k$, does there exist a collection of at most $k$ sets whose union is equal to $U$?
The Simple Reduction

- Transformation \( f \) maps \((G=(V,E),k)\) to \((U,S_1,\ldots,S_m,k')\)
  - \( U \leftarrow E \)
  - For each vertex \( v \in V \) create a set \( S_v \) containing all edges that touch \( v \)
  - \( k' \leftarrow k \)

- Reduction \( f \) is clearly polynomial-time to compute
- We need to prove that the resulting algorithm gives the right answer!
Proof of Correctness

- Two directions:
  - If the answer to Vertex-Cover on \((G,k)\) is YES then the answer for Set-Cover on \(f(G,k)\) is YES
    - If a set \(W\) of \(k\) vertices covers all edges then the collection \(\{S_v \mid v \in W\}\) of \(k\) sets covers all of \(U\)
  - If the answer to Set-Cover on \(f(G,k)\) is YES then the answer for Vertex-Cover on \((G,k)\) is YES
    - If a subcollection \(S_{v_1}, \ldots, S_{v_k}\) covers all of \(U\) then the set \(\{v_1, \ldots, v_k\}\) is a vertex cover in \(G\).
Problems we already know are NP-complete

- Circuit-SAT
- 3-SAT
- Independent-Set
- Clique
- Vertex-Cover
- Set-Cover
More NP-completeness

- Subset-Sum problem
  - Given $n$ integers $w_1, \ldots, w_n$ and integer $t$
  - Is there a subset of the $n$ input integers that adds up to exactly $t$?
3-SAT $\leq_P$ Subset-Sum

- Given a 3-CNF formula with $m$ clauses and $n$ variables
- Will create $2m+2n$ numbers that are $m+n$ digits long
  - Two numbers for each variable $x_i$
    - $t_i$ and $f_i$ (corresponding to $x_i$ being true or $x_i$ being false)
  - Two extra numbers for each clause
    - $u_j$ and $v_j$ (filler variables to handle number of false literals in clause $C_j$)
### 3-SAT \(\leq_p\) Subset-Sum

\[ C_3 = (x_1 \lor \neg x_2 \lor x_5) \]

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Graph Colorability

- **Defn:** Given a graph \( G=(V,E) \), and an integer \( k \), a \( k \)-coloring of \( G \) is
  - an assignment of up to \( k \) different colors to the vertices of \( G \) so that the endpoints of each edge have different colors.

- **3-Color:** Given a graph \( G=(V,E) \), does \( G \) have a 3-coloring?

- **Claim:** 3-Color is NP-complete

- **Proof:** 3-Color is in NP:
  - Hint is an assignment of red, green, blue to the vertices of \( G \)
  - Easy to check that each edge is colored correctly
3-SAT $\leq_p$ 3-Color

Reduction:
- We want to map a 3-CNF formula $F$ to a graph $G$ so that
  - $G$ is 3-colorable iff $F$ is satisfiable
3-SAT $\leq_p$ 3-Color

Base Triangle
3-SAT $\leq_p$ 3-Color

Variable Part:
in 3-coloring, variable colors correspond to some truth assignment (same color as T or F)
3-SAT $\leq_p$ 3-Color

Clause Part:
Add one 6 vertex gadget per clause connecting its ‘outer vertices’ to the literals in the clause...
Any truth assignment satisfying the formula can be extended to a 3-coloring of the graph.
3-SAT $\leq_p$ 3-Color

Any 3-coloring of the graph colors each gadget triangle using each color.
Any 3-coloring of the graph has an F opposite the O color in the triangle of each gadget.
Any 3-coloring of the graph has T at the other end of the blue edge connected to the F
Any 3-coloring of the graph yields a satisfying assignment to the formula.
Matching Problems

- **Perfect Bipartite Matching**
  - Given a bipartite graph $G = (V, E)$ where $V = X \sqcup Y$ and $E \subseteq X \times Y$, is there a set $M$ in $E$ such that every vertex in $V$ is in precisely one edge of $M$?

- **In P**
  - Network Flow gives $O(nm)$ algorithm where $n = |V|$, $m = |E|$.
3-Dimensional Matching

- Perfect Bipartite Matching is in P
  - Given a bipartite graph $G=(V,E)$ where $V=X \cup Y$ and $E \subseteq X \times Y$, is there a subset $M$ in $E$ such that every vertex in $V$ is in precisely one edge of $M$?

- 3-Dimensional Matching
  - Given a tripartite hypergraph $G=(V,E)$ where $V=X \cup Y \cup Z$ and $E \subseteq X \times Y \times Z$, is there a subset $M$ in $E$ such that every vertex in $V$ is in precisely one hyperedge of $M$?
    - is in NP: Certificate is the set $M$
3-Dimensional Matching

- **Theorem:** 3-Dimensional Matching is **NP-complete**
- **Proof:**
  - We’ve already seen that it is in **NP**
  - 3-Dimensional Matching is **NP-hard**:
    - Reduction from 3-SAT
    - Given a 3-CNF formula $F$ we create a tripartite hypergraph ("hyperedges" are triangles) $G$ based on $F$ as follows
3-SAT $\leq_p$ 3-Dimensional Matching

- Variable part:
  - If variable $x_i$ occurs $r_i$ times in $F$ create $r_i$ red and $r_i$ green triangles linked in a circle, one pair per occurrence
  - Perfect matching $M$ must either use all the green edges leaving red tips uncovered ($x_i$ is assigned false) or all the red edges leaving all green tips uncovered ($x_i$ is assigned true)
3-SAT \leq_p 3-Dimensional Matching

- **Clause part:** Two new nodes per clause joined to each of its literals:

\[ C_3 = (x_1 \lor \neg x_2 \lor x_5) \]
3-SAT $\leq_p$ 3-Dimensional Matching

- **Slack**: If there are $m$ clauses then there are $3m$ variable occurrences. That means $3m$ total tips are not covered by whichever of red or green triangles not chosen. Of these, $m$ are covered if each clause is satisfied. Need to cover the remaining $2m$ tips.

**Solution**: Add $2m$ pairs of slack vertices. Add triangles joining each pair with every tip!
3-SAT $\leq_p$ 3-Dimensional Matching

- **Well-formed:** Each triangle has one of each type of node:

- **Correctness:**
  - If $F$ has a satisfying assignment then choose the following triangles which form a perfect 3-dimensional matching in $G$:
    - Either the red or the green triangles in the cycle for $x_i$ - the opposite of the assignment to $x_i$.
    - The triangle containing the first true literal for each clause and the two clause nodes.
    - $2m$ slack triangles one per new pair of nodes to cover all the remaining tips.
3-SAT $\leq_p$ 3-Dimensional Matching

Correctness continued:

If $G$ has a perfect 3-dimensional matching then:

- Each blue node in the cycle for each $x_i$ is contained in exactly two triangles, exactly one of which must be in $M$. If one triangle in the cycle is in $M$, the others must be the same color. We use the color not used to define the truth assignment to $x_i$

- The two nodes for any clause must be contained in an edge which must also contain a third node that corresponds to a literal made true by the truth assignment. Therefore the truth assignment satisfies $F$ so it is satisfiable.