Lecture 19
The class NP

Definition:
\[ NP = \bigcup_{k \geq 1} \text{Nondeterministic-TIME}(n^k) \]
I.e., the set of (decision) problems solvable by computers in \textit{Nondeterministic} polynomial time. I.e., \( L \in NP \) iff there is a nondeterministic algorithm deciding \( L \) in time \( T(n) = O(n^k) \) for some fixed \( k \) (i.e., \( k \) is independent of the input).
Alternate Views of Nondeterminism

NTM – there is a path…

Parallel – make the tree

Search – look for a path (or sat-ing assignment or clique or…)

Guess and Check

Polynomial Verifier
Alternate Way To Define NP

A language $L$ is *polynomially verifiable* iff there is a polynomial time procedure $v(-,-)$, (the “verifier”) and an integer $k$ such that

- for every $x \in L$ there is a “hint” $h$ with $|h| \leq |x|^k$ such that $v(x,h) = \text{YES}$
- for every $x \notin L$ there is no hint $h$ with $|h| \leq |x|^k$ such that $v(x,h) = \text{YES}$

(“Hints,” sometimes called “certificates,” or “witnesses”, are just strings.)

Equivalently:

There is some integer $k$ and language $L_v$ in $P$ s.t.:

$$L = \{ x \mid \exists y, |y| \leq |x|^k \land \langle x,y \rangle \in L_v \}$$
Example: Clique

“Is there a k-clique in this graph?”

any subset of k vertices *might* be a clique

there are *many* such subsets, but I only need to find one

if I knew where it was, I could describe it succinctly, e.g.
"look at vertices 2,3,17,42,...",

I’d know one if I saw one: "yes, there are edges between
2 & 3, 2 & 17,... so it's a k-clique”

this can be quickly checked

And if there is *not* a k-clique, I wouldn’t be fooled by a
statement like “look at vertices 2,3,17,42,...”
More Formally: CLIQUE is in NP

procedure v(x,h)
    if
        x is a well-formed representation of a graph
        G = (V, E) and an integer k,
        and
        h is a well-formed representation of a k-vertex subset U of V,
        and
        U is a clique in G,
    then output "YES"
    else output "I'm unconvinced"
Is it correct?

For every $x = (G,k)$ such that $G$ contains a $k$-clique, there is a hint $h$ that will cause $v(x,h)$ to say YES, namely $h$ = a list of the vertices in such a $k$-clique and

No hint can fool $v$ into saying yes if either $x$ isn't well-formed (the uninteresting case) or if $x = (G,k)$ but $G$ does not have any cliques of size $k$ (the interesting case)
The 2 defns are equivalent

Theorem: $L$ in $NP$ iff $L$ is polynomially verifiable

Pf: $\Rightarrow$ Let $M$ be a poly time NTM for $L$, $x$ an input to $M$, $|x| = n$. If $x$ in $L$ there is an accepting computation history $y$ for $M$ on $x$. If $M$ runs $T = n^{O(1)}$ steps on $x$, then $y$ is $T+1$ configs, each of length $\sim T$, so $|y| = O(T^2) = n^{O(1)}$. Furthermore, a deterministic TM can check that $y$ is an accepting history of $M$ on $x$ in poly time. Critically, if $x$ is not accepted, no $y$ will pass this check. Thus, $L$ is poly time verifiable.

(We could equally well let $y$ encode the sequence of nondeterministic choices $M$ makes along some accepting path.)
The 2 defns are equivalent (cont.)

Theorem: L in NP iff L is polynomially verifiable

Pf: Suppose L is poly time verifiable, V is a time \( n^d \)-time TM implementing the verifier, and \( k \) is the exponent in the hint length bound. Consider this TM:

\[
M: \text{on input } x, \text{nondeterministically choose a string } y \text{ of length at most } |x|^k, \text{then run } V \text{ on } \langle x, y \rangle; \text{ accept iff it does.}
\]

Then M is an NTM accepting L: By defn of poly verifier \( x \in L \iff \exists y, |y| \leq |x|^k \land V \text{ accepts } \langle x, y \rangle, \text{ and M tries (nondeterministically) all such } y\text{'s, accepting iff it finds one that } V \text{ accepts.}

Time bound for M: \((|x| + |x|^k + 3)^d = O(n^{kd}) = n^{O(1)}\)
“Is there a satisfying assignment for this Boolean formula?”

any assignment might work
there are lots of them
I only need one
if I had one I could describe it succinctly, e.g., “x₁=T, x₂=F, ..., xₙ=T”
I'd know one if I saw one: "yes, plugging that in, I see formula = T...”
this can be quickly checked
And if the formula is unsatisfiable, I wouldn’t be fooled by , “x₁=T, x₂=F, ..., xₙ=F"
More Formally: SAT ∈ NP

Hint: the satisfying assignment A

Verifier: \( v(F,A) = \text{syntax}(F,A) \land \text{satisfies}(F,A) \)

Syntax: True iff F is a well-formed formula & A is a truth-assignment to its variables

Satisfies: plug A into F and evaluate

Correctness:

If F is satisfiable, it has some satisfying assignment A, and we’ll recognize it

If F is unsatisfiable, it doesn’t, and we won’t be fooled
Alternate Views of Nondeterminism

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Guess and Check

Polynomial Verifier
The complexity class NP

NP consists of all decision problems where

You can verify the YES answers efficiently (in polynomial time) given a short (polynomial-size) hint

And

No hint can fool your polynomial time verifier into saying YES for a NO instance

(implausible for all exponential time problems)
Keys to showing that a problem is in NP

What's the output? (must be YES/NO)
What's the input? Which are YES?
For every given YES input, is there a hint that would help? Is it polynomial length?
  OK if some inputs need no hint
For any given NO input, is there a hint that would trick you?
A_{TM} is in NP
Input: a pair <M,w>
Output: yes/no does M accept w
Hint: y, an accepting computation history of M on w
Clearly, such a y exists for all accepted x and only accepted x, so we accept the right x’s and reject the rest.
And it’s fast – checking successive configs in the history is at worst, quadratic in the length of the history, so the verifier for <x,y> runs in time |<x,y>|^O(1).
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