Lecture 8
re HW#1, Aeron says “If I made a comment, even if I didn't take off points *this* time, people should pay attention because I will take off points for the same mistake in the future...”
$EQ_{TM}$ is undecidable

$EQ_{TM} = \{ <M_1, M_2> \mid M_i \text{ are TMs s.t. } L(M_1) = L(M_2) \}$
**EQ\textsubscript{TM} is undecidable**

\[ EQ\textsubscript{TM} = \{ <M_1, M_2> \mid M_i \text{ are TMs s.t. } L(M_1) = L(M_2) \}\]

**Pf:** Will show \( \text{EMPTY}\textsubscript{TM} \leq_T EQ\textsubscript{TM} \)

Suppose \( EQ\textsubscript{TM} \) were decidable. Let \( M_\emptyset \) be a TM that accepts nothing, say one whose start state = \( q_{\text{reject}} \).

Consider the TM \( E \) that, given \( <M> \), builds \( <M, M_\emptyset> \), then calls the hypothetical subroutine for \( EQ\textsubscript{TM} \) on it, accepting/rejecting as it does. Now, \( <M, M_\emptyset> \in EQ\textsubscript{TM} \) if and only if \( M \) accepts \( \emptyset \), so, \( E \) decides whether \( M \in \text{EMPTY}\textsubscript{TM} \), which we know to be impossible. Contradiction
Linear Bounded Automata

Like a (1-tape) TM, but tape only long enough for input
(head stays put if try to move off either end of tape)

\[ M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}}) \]

\[ L(M) = \{ x \in \Sigma^* \mid M \text{ accepts } x \} \]
An Aside: The Chomsky Hierarchy

\[ \begin{array}{ll}
\text{TM} &= \text{phrase structure grammars} \\
\alpha A \beta &\rightarrow \alpha \gamma \beta \\
\text{LBA} &= \text{context-sensitive grammars} \\
\alpha A \beta &\rightarrow \alpha \gamma \beta, \ \gamma \neq \varepsilon \\
\text{PDA} &= \text{context-free grammars} \\
A &\rightarrow \gamma \\
\text{DFA} &= \text{regular grammars} \\
A &\rightarrow \text{abcB}
\end{array} \]
$A_{LBA}$ is decidable

$A_{LBA} = \{ <M, w> \mid M \text{ is an LBA and } w \in L(M) \}$

Key fact: the number of distinct configurations of an LBA on any input of length $n$ is bounded, namely

$$\leq n |Q| |\Gamma|^n$$

If $M$ runs for more than that many steps, it is looping

Decision procedure for $A_{LBA}$:

Simulate $M$ on $w$ and count steps; if it halts and accepts/rejects, do the same; if it exceeds that time bound, halt and reject.
$\text{EMPTY}_{\text{LBA}}$ is undecidable

Why is this hard, when the acceptance problem is not?

Loosely, it’s about infinitely many inputs, not just one
Can we exploit that, say to decide $A_{\text{TM}}$?
An idea. An LBA is a TM, so can it simulate M on w?
Only if M doesn’t use too much tape.
What about simulating M on w$\#$???????????????
Given M, build LBA M’ that, on input w # # # # ... #, simulates M on w, treating # as a blank. If M halts, do the same. If M tries to move off the right end of the tape, reject.

\[ L(M') = \{ w^{#^k} \mid M \text{ accepts } w \text{ using } \leq | w^{#^k} | \text{ tape cells} \} \]

Key point:
if M rejects w, M’ rejects w^{#^k} for all k, \( \therefore L(M') = \emptyset \)
if M accepts w, some k will be big enough, \( \therefore L(M') \neq \emptyset \)