Lecture 5
The Acceptance Problem for TMs

\[ A_{TM} = \{ <M, w> \mid M \text{ is a TM} \land w \in L(M) \} \]

Theorem: \( A_{TM} \) is Turing recognizable

Pf: It is recognized by a TM \( U \) that, on input \( <M, w> \), simulates \( M \) on \( w \) step by step. \( U \) accepts iff \( M \) does. \( \square \)

\( U \) is called a Universal Turing Machine
(Ancestor of the stored-program computer)

Note that \( U \) is a recognizer, not a decider.
Programming ENIAC, circa 1947

http://en.wikipedia.org/wiki/ENIAC
Cardinality

Two sets have equal cardinality if there is a bijection between them.

A set is *countable* if it is finite or has the same cardinality as the natural numbers.

Examples:
- $\Sigma^*$ is countable (think of strings as base-$|\Sigma|$ numerals).
- Even natural numbers are countable: $f(n) = 2n$.
- The Rationals are countable.
More cardinality facts

If \( f: A \to B \) in an injective function ("1-1", but not necessarily "onto"), then

\[
|A| \leq |B|
\]

(Intuitive: \( f \) is a bijection from \( A \) to its range, which is a subset of \( B \), and \( B \) can’t be smaller than a subset of itself.)

Theorem (Cantor-Schroeder-Bernstein):

If \( |A| \leq |B| \) and \( |B| \leq |A| \) then \( |A| = |B| \)
The Reals are Uncountable

Suppose they were
List them in order
Define \( X \) so that its \( i^{th} \) digit \( \neq \) \( i^{th} \) digit of \( i^{th} \) real
Then \( X \) is not in the list
Contradiction

A detail: avoid .000... , .9999... in \( X \)
Number of Languages in $\Sigma^*$ is Uncountable

Suppose they were
List them in order
Define $L$ so that $w_i \in L$ $\iff w_i \notin L_i$
Then $L$ is not in the list
Contradiction
“Most” languages are neither Turing recognizable nor Turing decidable

Pf:

“< >” maps TMs into $\Sigma^*$, a countable set, so the set of TMs, and hence of Turing recognizable languages is also countable; Turing decidable is a subset of Turing recognizable, so also countable. But by the previous result, the set of all languages is uncountable.
A specific non-Turing-recognizable language

Let $M_i$ be the TM encoded by $w_i$, i.e. $<M_i> = w_i$

$(M_i = \text{some default machine, if } w_i \text{ is an illegal code.})$

$i, j$ entry tells whether $M_i$ accepts $w_j$

Then $D$ is not recognized by any TM
Theorem: The class of Turing recognizable languages is \textit{not} closed under complementation.

Proof:

The \textit{complement} of $D$, is Turing recognizable:

On input $w_i$, run $<M_i>$ on $w_i$ (= $<M_i>$); accept if it does. E.g. use a universal TM on input $<M_i,<M_i>>$
Theorem: The class of Turing decidable languages is closed under complementation.

Proof:

Flip $q_{\text{accept}}, q_{\text{reject}}$
Decidable $\subseteq$ Recognizable

Diagram:

- Decidable
- Recognizable
- Co-recognizable

Relationships:

- Decidable $\subseteq$ Recognizable
- Decidable $\not\equiv$ Co-recognizable