Problem Set #6

Reading assignment: Sipser, Sections 7.1–7.5.

Instructions: Same as homework #1. This problem set has five regular problems worth 10 points each.

1. Show that if \( P = NP \), then every language \( A \in P \), except \( A = \emptyset \) and \( A = \Sigma^* \) is NP-complete.

2. Define the language
   \[
   U = \{ \langle M, x, 1^t \rangle : M \text{ is a nondeterministic Turing machine that accepts } x \text{ within } t \text{ steps} \}.
   \]
   Show that \( U \) is NP-complete.

3. A subset of the nodes of a graph \( G \) is a dominating set if every other node of \( G \) is adjacent to some node in the subset. Let
   \[
   \text{DOMINATING-SET} = \{ \langle G, k \rangle : G \text{ has a dominating set with } k \text{ nodes} \}.
   \]
   Show DOMINATING-SET is NP-complete by giving a reduction from VERTEX-COVER.

4. Let HALF-INDSET be the language
   \[
   \{ \langle G \rangle : G \text{ is an undirected graph that has an independent set of size at least } n/2 \},
   \]
   where \( n \) is the number of vertices in \( G \). Prove that INDSET \( \leq_p \) HALF-INDSET.

5. This problem investigates resolution, a method for proving the unsatisfiability of CNF-formulas. Let \( \phi = C_1 \land C_2 \land \cdots \land C_m \) be a formula in CNF, where the \( C_i \)'s are its clauses. Let \( \mathcal{C} = \{ C_i : C_i \text{ is a clause of } \phi \} \). In a resolution step, we take two clauses \( C_a, C_b \in \mathcal{C} \) which both have some variable \( x \), occurring positively in one of the clauses and negatively in the other. Thus \( C_a = (x \lor y_1 \lor y_2 \lor \cdots \lor y_k) \) and \( C_b = (\bar{x} \lor z_1 \lor z_2 \lor \cdots \lor z_{\ell}) \), where \( \{y_i\} \) and \( \{z_i\} \) are literals. We form the new clause \((y_1 \lor y_2 \lor \cdots \lor y_k \lor z_1 \lor z_2 \lor \cdots \lor z_{\ell})\) and remove repeated literals. Add this new clause to \( \mathcal{C} \). Repeat the resolution steps until no additional clauses can be obtained. If the empty clause (\( \)\) is in \( \mathcal{C} \) then declare \( \phi \) to be unsatisfiable.

   We say that resolution is sound if it never declares satisfiable formulas to be unsatisfiable. We say that resolution is complete if all unsatisfiable formulas are declared to be unsatisfiable.

   (a) Show that resolution is sound and complete.

   (b) Define \( 2\text{-SAT} = \{ \langle \phi \rangle : \phi \text{ is a satisfiable 2-CNF formula} \} \). (A 2-CNF formula is an AND of clauses where each clause is an OR of at most two literals.) Use part (a) to show that \( 2\text{-SAT} \in P \).