1. Show that there is an undecidable language $L \subseteq \{1\}^*$.

2. (Reduction of search problems to decision problems) Let $f : \Sigma_0^* \rightarrow \Sigma_0^*$ be an arbitrary function. Define a related language $L_f \subseteq \Sigma^*$ and describe a Turing machine to compute $f$ using a machine that decides $L_f$. Also show how to decide $L_f$ using a machine that computes $f$. The alphabet $\Sigma$ does not have to be the same as $\Sigma_0$.

3. Tell whether the following languages are (a) decidable, (b) recognizable but not decidable, (c) co-recognizable but not decidable, or (d) neither recognizable or co-recognizable. Justify your answer.
   
   (a) $\{\langle M \rangle : \text{TM } M \text{ halts within 2008 steps on some input}\}$.
   
   (b) $\{\langle M \rangle : \text{TM } M \text{ halts within 2008 steps on every input}\}$.

4. Let $\mathcal{P}$ be a collection of Turing-recognizable languages. Suppose that there exists an infinite language $L \in \mathcal{P}$ such that no finite subset of $L$ belongs to $\mathcal{P}$. In this case, prove that the language
   
   $\mathcal{P}_{TM} = \{\langle M \rangle : \text{M is a TM and } L(M) \in \mathcal{P}\}$
   
   is not Turing-recognizable.

5. **Extra credit.** Define a relation $R \subseteq (\Sigma^*)^k$ to be decidable if the language
   
   $L_R = \{\langle x_1, x_2, \ldots, x_k \rangle : (x_1, x_2, \ldots, x_k) \in R\}$
   
   is decidable. Define $\Sigma_k$ for $k \geq 0$ to be the class of all languages $L$ for which there is a decidable $(k + 1)$-ary relation $R$ such that
   
   $L = \{x : \exists x_1 \forall x_2 \exists x_3 \cdots Q_k x_k R(x_1, x_2, \ldots, x_k, x)\}$,
   
   where the quantifier $Q_k$ is $\exists$ if $k$ is odd and $\forall$ if $k$ is even. We define
   
   $\Pi_k = \text{co} \Sigma_k = \{L : \bar{L} \in \Sigma_k\}$.
   
   In this notation, $\Sigma_0$ is the set of decidable languages, and $\Sigma_1$ is the set of Turing-recognizable languages. Finally, define
   
   $\text{ALL}_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) = \Sigma^*\}$.
(a) Prove that $\text{ALL}_{TM}$ is $\Pi_2$-complete, in the sense that (i) it belongs to $\Pi_2$ and (ii) every language $A \in \Pi_2$ mapping reduces to $\text{ALL}_{TM}$.

(b) You know how to prove that $\text{ALL}_{TM} \notin \Pi_1$ (by proving that $A_{TM} \leq_m \text{ALL}_{TM}$). You don’t have to do this. Instead, use part (a) to show that $\text{ALL}_{TM} \notin \Sigma_1$, i.e. $\text{ALL}_{TM}$ is not Turing-recognizable.