Read the assignment: Chapters 4 and 5 of Sipser.

Instructions: Same as homework #1.

This problem set has four regular problems worth 10 points each, and one extra credit problem. Please be as careful as possible in your arguments and your answers.

1. Which of the following problems about Turing machines is decidable and which is not? Briefly justify your answers.
   
   (a) To determine, given a Turing machine $M$ and a string $w$, whether $M$ ever moves its head to the left when it is run on input $w$.
   
   (b) To determine, given a Turing machine $M$, whether the tape ever contains four consecutive 1’s during the course of $M$’s computation when it is run on input 01.

2. Let $A$ and $B$ be two disjoint languages. Say that $C$ separates $A$ and $B$ if $A \subseteq C$ and $B \subseteq \overline{C}$. Show that any two disjoint co-Turing-recognizable languages are separable by some decidable language.

3. (a) Prove that a language $A$ is Turing-recognizable if and only if $A$ is mapping reducible to $A_{TM}$.
   
   (b) Prove that a language $B$ is Turing-decidable if and only if $B$ is mapping reducible to $\{0^n1^n : n \geq 1\}$.

4. Define

   $$f(m) = \begin{cases} 
   3m + 1 & \text{for odd } m \\
   m/2 & \text{for even } m.
   \end{cases}$$

   for any natural number $m$. If you start with a number $m$ and iterate $f$, you obtain a sequence: $m, f(m), f(f(m)), \ldots$ Stop if you ever hit 1. Extensive computer tests have shown that every starting point $m$ between 1 and $10^{18}$ eventually gives a sequence that ends in 1. The question of whether this happens for all starting points is unsolved, and is called the $3m+1$ conjecture. Paul Erdős offered $500 for its solution. You don’t have to solve the conjecture.

   Suppose that $A_{TM}$ were decidable by a TM $H$. Use $H$ to describe a TM that is guaranteed to state the answer to the $3m+1$ conjecture.