## 1 In Class Exercise

Theorem 1. Let $G$ be a graph with $n$ vertices such that the degree of every vertex of $G$ is at most $k$. Prove that we can color vertices of $G$ with $k+1$ colors such that the endpoints of every edge get two distinct colors.

Proof This problem is a bit more complex because there are two parameters that we can induct on: $n$ and $k$. In this case, we let $k$ as a fixed number in the entire proof and we will prove the statement by induction on $n$.

We prove by induction on $n$. First define $P(n)$ be "every graph with $n$ vertices such that the degree of every vertex is at most $k$ can be colored with $k+1$ colors such that the endpoints of every edge have two distinct colors".
Base Case: $n=1$. In this case we color the single vertex with a color. We can do so because $k \geq 0$.
IH: Suppose $P(n-1)$ holds.
IS: We need to prove $P(n)$. Let $G$ be an arbitrary graph with $n$ vertices such that the degree of every vertex of $G$ is at most $k$. Let $v$ be an arbitrary vertex of $G$. Let $G^{\prime}=G-v$ (we also remove all edges incident to $v$. Now, by removing $v$ (and edges of $v$ ) we can only reduce degree of the rest of the vertices. Therefore, every vertex of $G^{\prime}$ also has degree at most $k$. Since $G^{\prime}$ has $n-1$ vertices by IH we can color vertices of $G^{\prime}$ with $k+1$ colors such that endpoints of every edge have distinct colors. Now, we color $G$. We color every vertex of $G$ (except $v$ ) with the same color in $G^{\prime}$. Now, to color $v$, note that it has at most $k$ neighbors. Since we have $k+1$ colors there is a color that is not used in any of the neighbors of $v$. We color $v$ with that color.

Note that this proof also gives an algorithm to color such a graph. Here is a sample execution of such an algorithm. Say $k=3$, so we have 4 colors available. Say we remove vertices in the following order $6,3,4,5,1$.


Now, we ca color. First, we color the last vertex 2 with blue. Then, we add back the removed vertices and each time we use a color not used on the neighbors: Note that to color the last vertex 6 we got lucky. Even though it had 3 neighbors, two of them were color blue. So, we could color 6 with green and this way totally we used only 3 colors (of 4 available colors). We also had the option of coloring 6 with orange and that would also be a valid coloring.


## 2 Coloring Planar graphs

Theorem 2. The vertices of any planar graph can be colored with 6 colors in such a way that every edge gets exactly two distinct colors.

In order to prove the theorem first prove the following claim:
Claim 3. In any planar graph there exists a vertex $v$ with $\operatorname{deg}(v) \leq 5$.
Proof of Claim 3: Hint: Feel free to use the following fact without proof:
Fact 4. For any planar graph with $n$ vertices and $m$ edges we have $3 n-4 \geq m$.
First, recall that for any graph $G$

$$
\sum_{v} \operatorname{deg}(v)=2 m .
$$

But since by claim assumption, $2 m \leq 6 n-8$, we have $\sum_{v} \operatorname{deg}(v) \leq 6 n-8$.
We prove by contradiction that there exists a vertex $v$ with $\operatorname{deg}(v) \leq 5$. If for all $v, \operatorname{deg}(v) \geq 6$, then

$$
6 n-4 \geq \sum_{v} \operatorname{deg}(v) \geq 6 n
$$

which is a contradiction.

## Proof of Thm 2:

Base Case: A planar graph with 1 vertex can be colored with 6 colors obviously.
IH: Every planar graph with $n-1$ vertices can be colored with 6 colors.
IS: We want to show that every planar graph with $n$ vertices can be colored with 6 colors. Let $G$ be a planar graph with $n$ vertices. We show that $G$ can be colored with 6 colors. By claim $G$ has a vertex $v$ with $\operatorname{deg}(v) \leq 5$. Let $H=G-\{v\}$.

We claim that $H$ is also planar. Because if we can draw $G$ on the plane with no crossing, when we remove $v$ and its edges, we still have a drawing of the remaining graph (i.e., $H$ ) with no crossing. Therefore, $H$ is a planar graph with $n-1$ vertices. So, by $\mathrm{IH}, H$ can be colored with 6 colors.

Now, let's add vertex $v$ (and its edges) back in. We need to find a consistent color vertex $v$ and this would complete the proof. By definition, $v$ has at most 5 neighbors. Since we have 6 colors, there exists a color which is not used in any of the neighbors of $v$. We color $v$ with that color and we obtain a consistent coloring.

