CSE 421: Introduction to Algorithms

Complexity and Representative Problems

Paul Beame
Measuring efficiency: The RAM model

- RAM = Random Access Machine

- Time ≈ # of instructions executed in an ideal assembly language
  - each simple operation (+, *, -, =, if, call) takes one time step
  - each memory access takes one time step
Complexity analysis

- Problem size $N$
  - Worst-case complexity: $\text{max} \ # \text{ steps}$
    algorithm takes on any input of size $N$
  - Best-case complexity: $\text{min} \ # \text{ steps}$
    algorithm takes on any input of size $N$
  - Average-case complexity: $\text{avg} \ # \text{ steps}$
    algorithm takes on inputs of size $N$
Stable Matching

- Problem size
  - \( N=2n^2 \) words
    - 2n people each with a preference list of length n
  - \( 2n^2 \log n \) bits
    - specifying an ordering for each preference list takes \( n \log n \) bits

- Brute force algorithm
  - Try all \( n! \) possible matchings

- Gale-Shapley Algorithm
  - \( n^2 \) iterations, each costing constant time
    - For each man an array listing the women in preference order
    - For each woman an array listing the preferences indexed by the names of the men
    - An array listing the current partner (if any) for each woman
    - An array listing the preference index of the last woman each man proposed to (if any)
Complexity

- The complexity of an algorithm associates a number \( T(N) \), the worst/average-case/best time the algorithm takes, with each problem size \( N \).

- Mathematically,
  - \( T \) is a function that maps positive integers giving problem size to positive real numbers giving number of steps.
Efficient = Polynomial Time

- **Polynomial time**
  - Running time $T(N) \leq cN^k + d$ for some $c, d, k \geq 0$

- **Why polynomial time?**
  - If problem size grows by at most a constant factor then so does the running time
    - E.g. $T(2N) \leq c(2N)^k + d \leq 2^k(cN^k + d)$
    - Polynomial-time is exactly the set of running times that have this property

- Typical running times are small degree polynomials, mostly less than $N^3$, at worst $N^6$, not $N^{100}$
Complexity

Problem size $N$

Time

$T(N)$

Problem size $N$
- Given two positive functions $f$ and $g$
  - $f(N)$ is $O(g(N))$ iff there is a constant $c > 0$ so that $f(N)$ is eventually always $\leq c \cdot g(N)$
  - $f(N)$ is $o(g(N))$ iff the ratio $f(N)/g(N)$ goes to 0 as $N$ gets large
  - $f(N)$ is $\Omega(g(N))$ iff there is a constant $\varepsilon > 0$ so that $f(N)$ is $\geq \varepsilon \cdot g(N)$ for infinitely many values of $N$
  - $f(N)$ is $\Theta(g(N))$ iff $f(N)$ is $O(g(N))$ and $f(N)$ is $\Omega(g(N))$

Note: The definition of $\Omega$ is the same as “$f(N)$ is not $o(g(N))$”
Complexity

Time

Problem size $N$

$T(N)$
Administrative

- Office hours set and on website:
  - Me: MWF 3:20-3:50, W 10-10:50
  - Xin: T 2:30-3:20, Th 11-11:50
  - Kuikui: Th 1:30-2:20

- Reading
  - Chapter 2, start Chapter 3
5 Representative Problems

- Interval Scheduling
  - Single resource
  - Reservation requests
  - Of form “Can I reserve it from start time $s$ to finish time $f$?”
    - $s < f$
Interval Scheduling

- **Input.** Set of jobs with start times and finish times.
- **Goal.** Find maximum cardinality subset of mutually compatible jobs.

Jobs don't overlap
Interval scheduling

- Formally
  - Requests 1, 2, ..., n
    - request \( i \) has start time \( s_i \) and finish time \( f_i > s_i \)
  
  - Requests \( i \) and \( j \) are compatible iff either
    - request \( i \) is for a time entirely before request \( j \)
      - \( f_i \leq s_j \)
    - or, request \( j \) is for a time entirely before request \( i \)
      - \( f_j \leq s_i \)
  
  - Set \( A \) of requests is compatible iff every pair of requests \( i, j \in A, i \neq j \) is compatible
  
  - Goal: Find maximum size subset \( A \) of compatible requests
Interval Scheduling

- We’ll see that an optimal solution can be found using a “greedy algorithm”
  - Myopic kind of algorithm that seems to have no look-ahead
  - These algorithms only work when the problem has a special kind of structure
  - When they do work they are typically very efficient
Weighted Interval Scheduling

- Same problem as interval scheduling except that each request $i$ also has an associated value or weight $w_i$

  - $w_i$ might be
    - amount of money we get from renting out the resource for that time period
    - amount of time the resource is being used
Weighted Interval Scheduling

- **Input.** Set of jobs with start times, finish times, and weights.
- **Goal.** Find maximum weight subset of mutually compatible jobs.

![Graph showing weighted interval scheduling](image)
**Weighted Interval Scheduling**

- Ordinary interval scheduling is a special case of this problem
  - Take all $w_i = 1$

- Problem is quite different though
  - E.g. one weight might dwarf all others

- “Greedy algorithms” don’t work

- **Solution**: “Dynamic Programming”
  - builds up optimal solutions from smaller problems using a compact table to store them
Bipartite Matching

- A graph $G = (V, E)$ is bipartite iff
  - $V$ consists of two disjoint pieces $X$ and $Y$ such that every edge $e$ in $E$ is of the form $(x, y)$ where $x \in X$ and $y \in Y$
  - Similar to stable matching situation but in that case all possible edges were present

- $M \subseteq E$ is a matching in $G$ iff no two edges in $M$ share a vertex
  - **Goal:** Find a matching $M$ in $G$ of maximum possible size
Bipartite Matching

- **Input.** Bipartite graph.
- **Goal.** Find maximum cardinality matching.
Bipartite Matching

- Models assignment problems
  - X represents jobs, Y represents machines
  - X represents professors, Y represents courses

- If |X|=|Y|=n
  - G has perfect matching iff maximum matching has size n

- Solution: polynomial-time algorithm using “augmentation” technique
  - also used for solving more general class of network flow problems
Independent Set

- Given a graph $G=(V,E)$
  - A set $I \subseteq V$ is independent iff no two nodes in $I$ are joined by an edge

- **Goal:** Find an independent subset $I$ in $G$ of maximum possible size

- Models conflicts and mutual exclusion
Independent Set

- **Input.** Graph.
- **Goal.** Find *maximum cardinality* independent set.
Independent Set

- Generalizes
  - Interval Scheduling
    - Vertices in the graph are the requests
    - Vertices are joined by an edge if they are not compatible

- Bipartite Matching
  - Given bipartite graph $G=\langle V, E \rangle$ create new graph $G'=\langle V', E' \rangle$ where
    - $V'=E$
    - Two elements of $V'$ (which are edges in $G$) are joined if they share an endpoint in $G$
Bipartite Matching vs Independent Set

\[ G = (U \cup V, E) \]

\[ G' = (V', E') \]
Independent Set

- No polynomial-time algorithm is known
  - But to convince someone that there was a large independent set all you’d need to do is show it to them
    - they can easily convince themselves that the set is large enough and independent
  - Convincing someone that there isn’t one seems much harder

- We will show that Independent Set is NP-complete
  - Class of all the hardest problems that have the property above
Competitive Facility Location

- Two players competing for market share in a geographic area
  - e.g. McDonald’s, Burger King
- Rules:
  - Region is divided into $n$ zones, $1, \ldots, n$
  - Each zone $i$ has a value $b_i$
    - Revenue derived from opening franchise in that zone
  - No adjacent zones may contain a franchise
    - i.e., zoning regulations limit density
  - Players alternate opening franchises
- Find: Given a target total value $B$ is there a strategy for the second player that always achieves $\geq B$?
Competitive Facility Location

- Model geography by
  - A graph $G = (V, E)$ where
    - $V$ is the set $\{1, \ldots, n\}$ of zones
    - $E$ is the set of pairs $(i, j)$ such that $i$ and $j$ are adjacent zones

- Observe:
  - The set of zones with franchises will form an independent set in $G$
Competitive Facility Location

Target $B = 20$ achievable?

What about $B = 25$?
Competitive Facility Location

- Checking that a strategy is good seems hard
  - You’d have to worry about all possible responses at each round!
    - a giant search tree of possibilities

- Problem is \textit{PSPACE-complete}
  - Likely strictly harder than \textit{NP-complete} problems
  - \textit{PSPACE-complete} problems include
    - Game-playing problems such as \( n \times n \) chess and checkers
    - Logic problems such as whether quantified boolean expressions are always true
    - Verification problems for finite automata
Five Representative Problems

- Variations on a theme: independent set.

- Interval scheduling: $O(n \log n)$ greedy algorithm.

- Weighted interval scheduling: $O(n \log n)$ dynamic programming algorithm.

- Bipartite matching: $O(n^k)$ max-flow based algorithm.

- Independent set: NP-complete.

- Competitive facility location: PSPACE-complete.