CSE 421
Algorithms
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Lecture 25
NP-Completeness
NP Completeness

I can't find an efficient algorithm, I guess I'm just too dumb.

I can't find an efficient algorithm, but neither can all these famous people.
Algorithms vs. Lower bounds

• Algorithmic Theory
  – What we can compute
    • I can solve problem X with resources R
  – Proofs are almost always to give an algorithm that meets the resource bounds

• Lower bounds
  – How do we show that something can’t be done?
Theory of NP Completeness
The Universe

- NP-Complete
- NP
- P
Polynomial Time

- **P**: Class of problems that can be solved in polynomial time
  - Corresponds with problems that can be solved efficiently in practice
  - Right class to work with “theoretically”
Decision Problems

• Theory developed in terms of yes/no problems
  – Independent set
    • Given a graph G and an integer K, does G have an independent set of size at least K
  – Network Flow
    • Given a graph G with edge capacities, a source vertex s, and sink vertex t, and an integer K, does the graph have flow function with value at least K
# Definition of P

Decision problems for which there is a polynomial time algorithm

<table>
<thead>
<tr>
<th>Problem</th>
<th>Description</th>
<th>Algorithm</th>
<th>Yes</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>MULTIPLE</td>
<td>Is x a multiple of y?</td>
<td>Grade school division</td>
<td>51, 17</td>
<td>51, 16</td>
</tr>
<tr>
<td>RELPRIME</td>
<td>Are x and y relatively prime?</td>
<td>Euclid’s algorithm</td>
<td>34, 39</td>
<td>34, 51</td>
</tr>
<tr>
<td>EDIT-DISTANCE</td>
<td>Is the edit distance between x and y less than 5?</td>
<td>Dynamic programming</td>
<td>neither</td>
<td>acgggt tttta</td>
</tr>
<tr>
<td>LSOLVE</td>
<td>Is there a vector x that satisfies Ax = b?</td>
<td>Gaussian elimination</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
What is NP?

• Problems solvable in non-deterministic polynomial time . . .

• Problems where “yes” instances have polynomial time checkable certificates
Certificate examples

• Independent set of size K
  – The Independent Set
• Satisfiable formula
  – Truth assignment to the variables
• Hamiltonian Circuit Problem
  – A cycle including all of the vertices
• K-coloring a graph
  – Assignment of colors to the vertices
Certifiers and Certificates: 3-Satisfiability

SAT: Does a given CNF formula have a satisfying formula

Certificate: An assignment of truth values to the n boolean variables

Certifier: Check that each clause has at least one true literal,

instance s

\[
\left( \overline{x_1} \lor x_2 \lor x_3 \right) \land \left( x_1 \lor \overline{x_2} \lor x_3 \right) \land \left( x_1 \lor x_2 \lor x_4 \right) \land \left( \overline{x_1} \lor \overline{x_3} \lor \overline{x_4} \right)
\]

certificate t

\[
x_1 = 1, \ x_2 = 1, \ x_3 = 0, \ x_4 = 1
\]
Certifiers and Certificates: Hamiltonian Cycle

HAM-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a simple cycle $C$ that visits every node?

Certificate. A permutation of the $n$ nodes.

Certifier. Check that the permutation contains each node in $V$ exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.
Polynomial time reductions

• Y is Polynomial Time Reducible to X
  – Solve problem Y with a polynomial number of computation steps and a polynomial number of calls to a black box that solves X
  – Notations: $Y \leq_p X$
Lemmas

• Suppose $Y \preceq_p X$. If $X$ can be solved in polynomial time, then $Y$ can be solved in polynomial time.

• Suppose $Y \preceq_p X$. If $Y$ cannot be solved in polynomial time, then $X$ cannot be solved in polynomial time.
NP-Completeness

- A problem $X$ is NP-complete if
  - $X$ is in NP
  - For every $Y$ in NP, $Y \leq_{p} X$

- $X$ is a “hardest” problem in NP

- If $X$ is NP-Complete, $Z$ is in NP and $X \leq_{p} Z$
  - Then $Z$ is NP-Complete
Cook’s Theorem

• The Circuit Satisfiability Problem is NP-Complete
Circuit SAT

Find a satisfying assignment
Garey and Johnson

COMPUTERS AND INTRACTABILITY
A Guide to the Theory of NP-Completeness

Michael R. Garey / David S. Johnson
History

- Jack Edmonds
  - Identified NP
- Steve Cook
  - Cook’s Theorem – NP-Completeness
- Dick Karp
  - Identified “standard” collection of NP-Complete Problems
- Leonid Levin
  - Independent discovery of NP-Completeness in USSR
P vs. NP Question

- NP-Complete
- NP
- P

P

NP
Populating the NP-Completeness Universe

- Circuit Sat $\leq_p$ 3-SAT
- 3-SAT $\leq_p$ Independent Set
- 3-SAT $\leq_p$ Vertex Cover
- Independent Set $\leq_p$ Clique
- 3-SAT $\leq_p$ Hamiltonian Circuit
- Hamiltonian Circuit $\leq_p$ Traveling Salesman
- 3-SAT $\leq_p$ Integer Linear Programming
- 3-SAT $\leq_p$ Graph Coloring
- 3-SAT $\leq_p$ Subset Sum
- Subset Sum $\leq_p$ Scheduling with Release times and deadlines
Sample Problems

• Independent Set
  – Graph $G = (V, E)$, a subset $S$ of the vertices is independent if there are no edges between vertices in $S$
Vertex Cover

- **Vertex Cover**
  - Graph $G = (V, E)$, a subset $S$ of the vertices is a vertex cover if every edge in $E$ has at least one endpoint in $S$
Cook’s Theorem

• The Circuit Satisfiability Problem is NP-Complete

• Circuit Satisfiability
  – Given a boolean circuit, determine if there is an assignment of boolean values to the input to make the output true
Circuit SAT

Find a satisfying assignment
Proof of Cook’s Theorem

• Reduce an arbitrary problem Y in NP to X
• Let A be a non-deterministic polynomial time algorithm for Y
• Convert A to a circuit, so that Y is a Yes instance iff and only if the circuit is satisfiable
Populating the NP-Completeness Universe

- Circuit Sat $\leq_P$ 3-SAT
- 3-SAT $\leq_P$ Independent Set
- 3-SAT $\leq_P$ Vertex Cover
- Independent Set $\leq_P$ Clique
- 3-SAT $\leq_P$ Hamiltonian Circuit
- Hamiltonian Circuit $\leq_P$ Traveling Salesman
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Satisfiability

Literal: A Boolean variable or its negation. \( x_i \) or \( \overline{x_i} \)

Clause: A disjunction of literals. \( C_j = x_1 \lor \overline{x_2} \lor x_3 \)

Conjunctive normal form: A propositional formula \( \Phi \) that is the conjunction of clauses. \( \Phi = C_1 \land C_2 \land C_3 \land C_4 \)

SAT: Given CNF formula \( \Phi \), does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

Ex: \( (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_3) \land (x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3}) \)

Yes: \( x_1 = \text{true}, \ x_2 = \text{true} \ x_3 = \text{false.} \)
3-SAT is NP-Complete

Theorem. 3-SAT is NP-complete.

Pf. Suffices to show that CIRCUIT-SAT \( \leq_p \) 3-SAT since 3-SAT is in NP.
  
  – Let K be any circuit.
  
  – Create a 3-SAT variable \( x_i \) for each circuit element \( i \).
  
  – Make circuit compute correct values at each node:
    
    • \( x_2 = \neg x_3 \Rightarrow \) add 2 clauses: \( x_2 \lor \neg x_3, \ \neg x_2 \lor \neg x_3 \)
    • \( x_1 = x_4 \lor x_5 \Rightarrow \) add 3 clauses: \( x_1 \lor x_4, \ x_1 \lor x_5, \ x_1 \lor x_4 \lor x_5 \)
    • \( x_0 = x_1 \land x_2 \Rightarrow \) add 3 clauses: \( \neg x_0 \lor x_1, \ \neg x_0 \lor x_2, \ x_0 \lor \neg x_1 \lor \neg x_2 \)
  
  – Hard-coded input values and output value.
    
    • \( x_5 = 0 \Rightarrow \) add 1 clause: \( \neg x_5 \)
    • \( x_0 = 1 \Rightarrow \) add 1 clause: \( x_0 \)
  
  – Final step: turn clauses of length < 3 into clauses of length exactly 3. •
3 Satisfiability Reduces to Independent Set

Claim. 3-SAT ≤ₚ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

– G contains 3 vertices for each clause, one for each literal.
– Connect 3 literals in a clause in a triangle.
– Connect literal to each of its negations.

\[ \Phi = (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor x_2 \lor x_4) \]
3 Satisfiability Reduces to Independent Set

Claim. G contains independent set of size $k = |\Phi|$ iff $\Phi$ is satisfiable.

Pf. $\Rightarrow$ Let $S$ be independent set of size $k$.
   - $S$ must contain exactly one vertex in each triangle.
   - Set these literals to true. and any other variables in a consistent way
   - Truth assignment is consistent and all clauses are satisfied.

Pf $\Leftarrow$ Given satisfying assignment, select one true literal from each triangle. This is an independent set of size $k$. ▪

\[
\Phi = (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor x_2 \lor x_4)
\]
IS \leq_P VC

• Lemma: A set S is independent iff V-S is a vertex cover

• To reduce IS to VC, we show that we can determine if a graph has an independent set of size K by testing for a Vertex cover of size n - K
IS $\leq_p$ VC

Find a maximum independent set S

Show that V-S is a vertex cover
Clique

- Clique
  - Graph $G = (V, E)$, a subset $S$ of the vertices is a clique if there is an edge between every pair of vertices in $S$
Complement of a Graph

- Defn: $G' = (V, E')$ is the complement of $G = (V, E)$ if $(u, v)$ is in $E'$ iff $(u, v)$ is not in $E$
IS $\leq_p$ Clique

- Lemma: S is Independent in G iff S is a Clique in the complement of G

- To reduce IS to Clique, we compute the complement of the graph. The complement has a clique of size K iff the original graph has an independent set of size K
Hamiltonian Circuit Problem

- Hamiltonian Circuit – a simple cycle including all the vertices of the graph
Thm: Hamiltonian Circuit is NP Complete

• Reduction from 3-SAT
Traveling Salesman Problem

• Given a complete graph with edge weights, determine the shortest tour that includes all of the vertices (visit each vertex exactly once, and get back to the starting point)

Find the minimum cost tour
Thm: $HC \leq_p TSP$
Graph Coloring

- NP-Complete
  - Graph K-coloring
  - Graph 3-coloring

- Polynomial
  - Graph 2-Coloring