CSE 421
Algorithms
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Lecture 27
NP-Completeness Proofs

NP-Completeness

- A problem X is NP-complete if
  - X is in NP
  - For every Y in NP, Y ≤p X

Cook’s Theorem

- The Circuit Satisfiability Problem is NP-Complete
- Proof ideas
  - Let A be an arbitrary problem in NP
  - Show that an instance x of A can be transformed in polynomial time into an instance y of Circuit SAT, such that x is a true instance of A iff y is a satisfiable circuit
  - A ≤p Circuit SAT

Populating the NP-Completeness Universe

- Circuit SAT ≤p 3-SAT
- 3-SAT ≤p Independent Set
- 3-SAT ≤p Vertex Cover
- Independent Set ≤p Clique
- 3-SAT ≤p Hamiltonian Circuit
- Hamiltonian Circuit ≤p Traveling Salesman
- 3-SAT ≤p Integer Linear Programming
- 3-SAT ≤p Graph Coloring
- 3-SAT ≤p Subset Sum
- Subset Sum ≤p Scheduling with Release times and deadlines

Satisfiability

Literal: A Boolean variable or its negation.

\( x_i \) or \( \overline{x_i} \)

Clause: A disjunction of literals.

\( C_i = x_i \lor \overline{x_i} \lor x_j \)

Conjunctive normal form: A propositional formula \( \Phi \) that is the conjunction of clauses.

\( \Phi = C_1 \land C_2 \land C_3 \land C_4 \)

SAT: Given CNF formula \( \Phi \), does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

\[
\text{Ex: } (x_1 \lor x_2 \lor x_3) \land (x_4 \lor x_5 \lor x_6) \land (x_7 \lor x_8 \lor x_9) \land (x_{10} \lor x_{11} \lor x_{12}) \]

Yes: \( x_1 = \text{true}, x_2 = \text{true}, x_3 = \text{false} \)

3-SAT is NP-Complete

Theorem. 3-SAT is NP-complete.

Proof. Suffices to show that CIRCUIT-SAT ≤p 3-SAT since 3-SAT is in NP.

- Let \( K \) be any circuit.
- Create a 3-SAT variable \( x_i \) for each circuit element \( i \).
- Make circuit compute correct values at each node:
  - \( x_i = x_j \) \( \Rightarrow \) add 2 clauses:
    - \( x_i = x_j \lor \overline{x_j} \)
    - \( x_i = x_j \lor x_j \)
  - \( x_i = x_j \lor \overline{x_k} \) \( \Rightarrow \) add 3 clauses:
    - \( x_i = x_j \lor \overline{x_k} \lor \overline{x_j} \)
    - \( x_i = x_j \lor x_j \lor \overline{x_k} \)
    - \( x_i = x_j \lor \overline{x_k} \lor x_j \)
- Hard-coded input values and output value.
  - \( x_1 = 0 \) \( \Rightarrow \) add 1 clause: \( \overline{x_1} \)
  - \( x_1 = 1 \) \( \Rightarrow \) add 1 clause: \( x_1 \)
- Final step: turn clauses of length < 3 into clauses of length exactly 3.
Proving a problem A is NP Complete

• Show A is in NP (usually easy)
• Choose an NP complete problem B
  – Convert an instance of B into an equivalent instance of A

3 Satisfiability Reduces to Independent Set

Claim. 3-SAT \leq_P INDEPENDENT-SET.
Pf. Given an instance \( \Phi \) of 3-SAT, we construct an instance \((G, k)\) of INDEPENDENT-SET that has an independent set of size \( k \) if \( \Phi \) is satisfiable.

Construction.
– \( G \) contains 3 vertices for each clause, one for each literal.
– Connect 3 literals in a clause in a triangle.
– Connect literal to each of its negations.

Analysis of 3-SAT to IS reduction

• Truth setting
  – \( X \) is true if at least one \( X \) is in the independent set
  – \( \overline{X} \) is false if at least one \( \overline{X} \) is in the independent set
• Truth consistency
  – Edges between all copies of \( X \) and \( \overline{X} \) ensure variables are true or false

IS \leq_P VC

• Lemma: A set \( S \) is independent iff \( V-S \) is a vertex cover
  • To reduce IS to VC, we show that we can determine if a graph has an independent set of size \( K \) by testing for a Vertex cover of size \( n - K \)
IS $\leq_P VC$

Find a maximum independent set $S$

Show that $V - S$ is a vertex cover

Clique

- Clique
  - Graph $G = (V, E)$, a subset $S$ of the vertices is a clique if there is an edge between every pair of vertices in $S$

Complement of a Graph

- Defn: $G' = (V, E')$ is the complement of $G = (V, E)$ if $(u, v)$ is in $E'$ iff $(u, v)$ is not in $E$

IS $\leq_P$ Clique

- Lemma: $S$ is Independent in $G$ iff $S$ is a Clique in the complement of $G$

  - To reduce IS to Clique, we compute the complement of the graph. The complement has a clique of size $K$ iff the original graph has an independent set of size $K$

Hamiltonian Circuit Problem

- Hamiltonian Circuit – a simple cycle including all the vertices of the graph

Thm: Hamiltonian Circuit is NP Complete

- Reduction from 3-SAT
Hamiltonian Path

- Is there a simple path that visits all the vertices?
- Is there a simple path from s to t that visits all the vertices?

Reduce HC to HP

\[ G_2 \text{ has a Hamiltonian Path if } G_1 \text{ has a Hamiltonian Circuit} \]

Traveling Salesman Problem

- Given a complete graph with edge weights, determine the shortest tour that includes all of the vertices (visit each vertex exactly once, and get back to the starting point)

Thm: HC \text{ } <_p \text{ } TSP

Graph Coloring

- NP-Complete
  - Graph K-coloring
  - Graph 3-coloring
- Polynomial
  - Graph 2-Coloring