CSE 421
Algorithms
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Lecture 24
Network Flow Applications
Today’s topics

• Network flow reductions
  – Multi source flow
  – Reviewer Assignment
• Baseball Scheduling
• Image Segmentation
• Strip Mining
• Reading: 7.5, 7.6, 7.10-7.12
Multi-source network flow

• Multi-source network flow
  – Sources $s_1, s_2, \ldots, s_k$
  – Sinks $t_1, t_2, \ldots, t_j$

• Solve with Single source network flow
Bipartite Matching

• A graph $G=(V,E)$ is bipartite if the vertices can be partitioned into disjoint sets $X, Y$

• A matching $M$ is a subset of the edges that does not share any vertices

• Find a matching as large as possible
Converting Matching to Network Flow
Resource Allocation:
Assignment of reviewers

- A set of papers $P_1, \ldots, P_n$
- A set of reviewers $R_1, \ldots, R_m$
- Paper $P_i$ requires $A_i$ reviewers
- Reviewer $R_j$ can review $B_j$ papers
- For each reviewer $R_j$, there is a list of paper $L_{j1}, \ldots, L_{jk}$ that $R_j$ is qualified to review
Baseball elimination

• Can the Dinosaurs win the league?

• Remaining games:
  – AB, AC, AD, AD, AD, BC, BC, BC, BD, CD

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A team wins the league if it has strictly more wins than any other team at the end of the season.
A team ties for first place if no team has more wins, and there is some other team with the same number of wins.
Baseball elimination

• Can the Fruit Flies win or tie the league?

• Remaining games:

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Assume Fruit Flies win remaining games

• Fruit Flies are tied for first place if no team wins more than 19 games

• Allowable wins
  – Ants (2)
  – Bees (3)
  – Cockroaches (3)
  – Dinosaurs (5)
  – Earthworms (5)

• 18 games to play

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Remaining games

Minimum Cut Applications

• Image Segmentation
• Open Pit Mining / Task Selection Problem
• Reduction to Min Cut problem

S, T is a cut if S, T is a partition of the vertices with \( s \) in S and \( t \) in T
The capacity of an S, T cut is the sum of the capacities of all edges going from S to T
Image Segmentation

- Separate foreground from background
Image analysis

- $a_i$: value of assigning pixel $i$ to the foreground
- $b_i$: value of assigning pixel $i$ to the background
- $p_{ij}$: penalty for assigning $i$ to the foreground, $j$ to the background or vice versa
- $A$: foreground, $B$: background
- $Q(A,B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{(i,j) \in E, i \in A, j \in B} p_{ij}$
Pixel graph to flow graph
Mincut Construction

\[ s \]

\[ a_v \]

\[ p_{vu} \]

\[ u \]

\[ p_{uv} \]

\[ v \]

\[ b_v \]

\[ t \]
Open Pit Mining
Application of Min-cut

- Open Pit Mining Problem
- Task Selection Problem
- Reduction to Min Cut problem

S, T is a cut if S, T is a partition of the vertices with s in S and t in T
The capacity of an S, T cut is the sum of the capacities of all edges going from S to T
Open Pit Mining

• Each unit of earth has a profit (possibly negative)
• Getting to the ore below the surface requires removing the dirt above
• Test drilling gives reasonable estimates of costs
• Plan an optimal mining operation
Mine Graph
Determine an optimal mine
Generalization

- Precedence graph $G=(V,E)$
- Each $v$ in $V$ has a profit $p(v)$
- A set $F$ is *feasible* if when $w$ in $F$, and $(v,w)$ in $E$, then $v$ in $F$.
- Find a feasible set to maximize the profit
Min cut algorithm for profit maximization

• Construct a flow graph where the minimum cut identifies a feasible set that maximizes profit
Precedence graph construction

- Precedence graph $G=(V,E)$
- Each edge in $E$ has infinite capacity
- Add vertices $s$, $t$
- Each vertex in $V$ is attached to $s$ and $t$ with finite capacity edges
Find a finite value cut with at least two vertices on each side of the cut.
The sink side of a finite cut is a feasible set

• No edges permitted from S to T
• If a vertex is in T, all of its ancestors are in T
Setting the costs

- If $p(v) > 0$,
  - $\text{cap}(v,t) = p(v)$
  - $\text{cap}(s,v) = 0$
- If $p(v) < 0$
  - $\text{cap}(s,v) = -p(v)$
  - $\text{cap}(v,t) = 0$
- If $p(v) = 0$
  - $\text{cap}(s,v) = 0$
  - $\text{cap}(v,t) = 0$
Minimum cut gives optimal solution
Why?
Computing the Profit

- Cost(W) = \(\sum_{w \in W; p(w) < 0} p(w)\)
- Benefit(W) = \(\sum_{w \in W; p(w) > 0} p(w)\)
- Profit(W) = Benefit(W) – Cost(W)

- Maximum cost and benefit
  - \(C = \text{Cost}(V)\)
  - \(B = \text{Benefit}(V)\)
Express Cap(S,T) in terms of B, C, Cost(T), Benefit(T), and Profit(T)

\[ \text{Cap}(S,T) = \text{Cost}(T) + \text{Ben}(S) = \text{Cost}(T) + \text{Ben}(S) + \text{Ben}(T) - \text{Ben}(T) \]
\[ = B + \text{Cost}(T) - \text{Ben}(T) = B - \text{Profit}(T) \]