Announcements

• Reading
  – Chapter 2.1, 2.2
  – Chapter 3 (Mostly review)
  – Start on Chapter 4

• Homework Guidelines
  – Prove that your algorithm works
    • A proof is a “convincing argument”
  – Give the run time for you algorithm
    • Justify that the algorithm satisfies the runtime bound
  – You may lose points for style
What does it mean for an algorithm to be efficient?
Definitions of efficiency

• Fast in practice

• Qualitatively better worst case performance than a brute force algorithm
Polynomial time efficiency

• An algorithm is efficient if it has a polynomial run time

• Run time as a function of problem size
  – Run time: count number of instructions executed on an underlying model of computation
  – $T(n)$: maximum run time for all problems of size at most $n$
Polynomial Time

- Algorithms with polynomial run time have the property that increasing the problem size by a constant factor increases the run time by at most a constant factor (depending on the algorithm)
Why Polynomial Time?

• Generally, polynomial time seems to capture the algorithms which are efficient in practice

• The class of polynomial time algorithms has many good, mathematical properties
Polynomial vs. Exponential Complexity

• Suppose you have an algorithm which takes n! steps on a problem of size n

• If the algorithm takes one second for a problem of size 10, estimate the run time for the following problems sizes:

  12  14  16  18  20
Ignoring constant factors

• Express run time as $O(f(n))$
• Emphasize algorithms with slower growth rates
• Fundamental idea in the study of algorithms
• Basis of Tarjan/Hopcroft Turing Award
Why ignore constant factors?

• Constant factors are arbitrary
  – Depend on the implementation
  – Depend on the details of the model

• Determining the constant factors is tedious and provides little insight
Why emphasize growth rates?

• The algorithm with the lower growth rate will be faster for all but a finite number of cases
• Performance is most important for larger problem size
• As memory prices continue to fall, bigger problem sizes become feasible
• Improving growth rate often requires new techniques
Formalizing growth rates

- $T(n)$ is $O(f(n))$ \quad \text{[}T : \mathbb{Z}^+ \rightarrow \mathbb{R}^+\text{]}
  - If $n$ is sufficiently large, $T(n)$ is bounded by a constant multiple of $f(n)$
  - Exist $c$, $n_0$, such that for $n > n_0$, $T(n) < c f(n)$

- $T(n)$ is $O(f(n))$ will be written as: $T(n) = O(f(n))$
  - Be careful with this notation
Prove $3n^2 + 5n + 20$ is $O(n^2)$

Let $c =$

Let $n_0 =$

$T(n)$ is $O(f(n))$ if there exist $c, n_0$, such that for $n > n_0$, $T(n) < c f(n)$
Order the following functions in increasing order by their growth rate

a) \( n \log^4 n \)
b) \( 2n^2 + 10n \)
c) \( 2^{n/100} \)
d) \( 1000n + \log^8 n \)
e) \( n^{100} \)
f) \( 3^n \)
g) \( 1000 \log^{10} n \)
h) \( n^{1/2} \)
Lower bounds

• $T(n)$ is $\Omega(f(n))$
  - $T(n)$ is at least a constant multiple of $f(n)$
  - There exists an $n_0$, and $\varepsilon > 0$ such that $T(n) > \varepsilon f(n)$ for all $n > n_0$

• Warning: definitions of $\Omega$ vary

• $T(n)$ is $\Theta(f(n))$ if $T(n)$ is $O(f(n))$ and $T(n)$ is $\Omega(f(n))$
Useful Theorems

• If \( \lim (f(n) / g(n)) = c \) for \( c > 0 \) then \( f(n) = \Theta(g(n)) \)

• If \( f(n) \) is \( O(g(n)) \) and \( g(n) \) is \( O(h(n)) \) then \( f(n) \) is \( O(h(n)) \)

• If \( f(n) \) is \( O(h(n)) \) and \( g(n) \) is \( O(h(n)) \) then \( f(n) + g(n) \) is \( O(h(n)) \)
Ordering growth rates

• For $b > 1$ and $x > 0$
  – $\log^b n$ is $O(n^x)$

• For $r > 1$ and $d > 0$
  – $n^d$ is $O(r^n)$
Graph Theory

- $G = (V, E)$
  - $V$ – vertices
  - $E$ – edges
- Undirected graphs
  - Edges sets of two vertices $\{u, v\}$
- Directed graphs
  - Edges ordered pairs $(u, v)$
- Many other flavors
  - Edge / vertices weights
  - Parallel edges
  - Self loops
Definitions

• Path: $v_1, v_2, \ldots, v_k$, with $(v_i, v_{i+1})$ in $E$
  – Simple Path
  – Cycle
  – Simple Cycle

• Distance

• Connectivity
  – Undirected
  – Directed (strong connectivity)

• Trees
  – Rooted
  – Unrooted
Graph search

• Find a path from $s$ to $t$

$S = \{s\}$

While there exists $(u, v)$ in $E$ with $u$ in $S$ and $v$ not in $S$

$\text{Pred}[v] = u$

Add $v$ to $S$

if $(v = t)$ then path found
Breadth first search

• Explore vertices in layers
  – s in layer 1
  – Neighbors of s in layer 2
  – Neighbors of layer 2 in layer 3 . . .
Key observation

• All edges go between vertices on the same layer or adjacent layers
Bipartite Graphs

- A graph $V$ is bipartite if $V$ can be partitioned into $V_1$, $V_2$ such that all edges go between $V_1$ and $V_2$
- A graph is bipartite if it can be two colored
Can this graph be two colored?
Algorithm

- Run BFS
- Color odd layers red, even layers blue
- If no edges between the same layer, the graph is bipartite
- If edge between two vertices of the same layer, then there is an odd cycle, and the graph is not bipartite
Theorem: A graph is bipartite if and only if it has no odd cycles
Lemma 1

- If a graph contains an odd cycle, it is not bipartite
Lemma 2

• If a BFS tree has an *intra-level edge*, then the graph has an odd length cycle

Intra-level edge: both end points are in the same level
Lemma 3

• If a graph has no odd length cycles, then it is bipartite