CSE 421: Intro Algorithms

2: Analysis

Larry Ruzzo
Why big-O: measuring algorithm efficiency
What’s big-O: definition and related concepts
Reasoning with big-O: examples & applications
   polynomials
   exponentials
   logarithms
   sums
Polynomial Time
Why big-O: measuring algorithm efficiency
Our correct TSP algorithm was incredibly slow. Basically slow no matter what computer you have. As a 2nd example, for large problems, quicksort is much faster than insertion sort, again no matter what computer you have – $n \log n$ vs $n^2$ matters a lot.

We want a general theory of “efficiency” that is

- Simple
- Objective
- Relatively independent of changing technology
- But still *predictive* – “theoretically bad” algorithms should be bad in practice and vice versa (usually)
“Runs fast on typical real problem instances”

Pro:
- sensible, bottom-line-oriented

Con:
- moving target (diff computers, compilers, Moore’s law)
- highly subjective (how fast is “fast”? What’s “typical”?)
defining efficiency

“Runs fast on a specific suite of benchmarks”

Pro:
again sensible, bottom-line-oriented

Con:
all the problems above
are benchmarks representative
algorithms can be “tuned” to the well-known benchmarks
generating/maintaining benchmarks is a burden
benchmarking a new algorithm is a lot of work
defining efficiency

Instead:

a) Give up on detailed timing, focus on scaling
   Nanoseconds matter of course, but we often want to push to bigger problems tomorrow than we can solve today, so an algorithm that scales as \( n^2 \), say, will very likely beat one that grows as \( 2^n \) or \( n^{10} \) or even \( n^3 \), even if the later uses fewer nanoseconds for today’s \( n \).

b) Give up on “typical,” focus on worst case behavior
   Over all inputs of size \( n \), how fast are we on the worst? Removes all debate about “typical” / “average.”

Overall, these yield a big win in terms of technology independence, ease of analysis, robustness
The *time complexity* of an algorithm associates a number $T(n)$, the *worst-case time* the algorithm takes, with each *problem size* $n$.

Mathematically,

$$T: \mathbb{N}^+ \rightarrow \mathbb{R}$$

i.e., $T$ is a function mapping positive integers (problem sizes) to positive real numbers (number of steps).

“*Reals*” so, e.g., we can say $\sqrt{n}$ instead of $\lceil \sqrt{n} \rceil$

“*Positive*” so, e.g., $\log(n)$ and $2^n/n$ aren’t problematic
computational complexity
why worst-case analysis?

Appropriate for time-critical applications
  E.g. avionics, nuclear reactors
Unlike Average-Case, no debate about what the right definition is
  If worst $\gg$ average, then (a) alg is doing something pretty subtle, & (b) are hard instances really that rare?
Analysis often much easier
Result is often representative of “typical” problem instances
Of course there are exceptions…
computational complexity: general goals

Asymptotic growth rate, i.e., characterize growth rate of worst-case run time as a function of problem size, up to a constant factor, e.g. \( T(n) = \mathcal{O}(n^2) \)

Why not try to be more precise?

- Average-case, e.g., is hard to define, analyze
- Technological variations (computer, compiler, OS, …) easily 10x or more
- Being more precise is much more work

A key question is “scale up”: if I can afford this today, how much longer will it take when my business is 2x larger? (E.g. today: \( cn^2 \), next year: \( c(2n)^2 = 4cn^2 \): 4 x longer.) Big-O analysis is adequate to address this.
What’s big-O: definition and related concepts
Given two functions $f$ and $g$: $\mathbb{N}^+ \to \mathbb{R}$

- $f(n)$ is $O(g(n))$ iff there is a constant $c > 0$ so that $f(n)$ is eventually always $\leq c \cdot g(n)$

- $f(n)$ is $\Omega(g(n))$ iff there is a constant $c > 0$ so that $f(n)$ is eventually always $\geq c \cdot g(n)$

- $f(n)$ is $\Theta(g(n))$ iff there are constants $c_1, c_2 > 0$ so that eventually always $c_1 g(n) \leq f(n) \leq c_2 g(n)$

“Eventually always $P(n)$” means “$\exists n_0$ s.t. $\forall n > n_0$ $P(n)$ is true.” I.e., there can be exceptions, but only for finitely many “small” values of $n$. 

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Reasoning with big-O: examples & applications

- polynomials
- exponentials
- logarithms
- sums
Show $10n^2 - 16n + 100$ is $O(n^2)$:

$10n^2 - 16n + 100 \leq 10n^2 + 100$

$= 10n^2 + 10^2$

$\leq 10n^2 + n^2 = 11n^2$ for all $n \geq 10$

$\therefore O(n^2)$ [and also $O(n^3), O(n^4), O(n^{2.5}), \ldots$]
Show $10n^2-16n+100$ is $\Omega(n^2)$:

$10n^2-16n+100 \geq 10n^2 - 16n$

\[ \geq 10n^2 - n^2 = 9n^2 \text{ for all } n \geq 16 \]

$\therefore \Omega(n^2)$ [ and also $\Omega(n)$, $\Omega(n^{1.5})$, … ]

Therefore also $10n^2-16n+100$ is $\Theta(n^2)$
[but not $\Theta(n^{1.999})$ or $\Theta(n^{2.001})$ ]
asymptotic bounds for polynomials

Polynomials:
\[ p(n) = a_0 + a_1 n + \ldots + a_d n^d \] is \( \Theta(n^d) \) if \( a_d > 0 \)

Proof:
\[
p(n) = a_0 + a_1 n + \ldots + a_d n^d \\
\leq |a_0| + |a_1| n + \ldots + a_d n^d \\
\leq |a_0|^d + |a_1|^d n^d + \ldots + a_d n^d \quad \text{(for } n \geq 1) \\
= c n^d, \text{ where } c = (|a_0| + |a_1| + \ldots + |a_{d-1}| + a_d) \\
\therefore \quad p(n) = O(n^d)
\]

Exercise: show that \( p(n) = \Omega(n^d) \)

Hint: this direction is trickier; focus on the “worst case” where all coefficients except \( a_d \) are negative.
another example of working with $O$-$\Omega$-$\Theta$ notation

Example: For any $a$, and any $b > 0$, $(n+a)^b$ is $\Theta(n^b)$

$(n+a)^b \leq (2n)^b \quad \text{for} \quad n \geq |a|$

$= 2^b n^b$

$= cn^b \quad \text{for} \quad c = 2^b$

so $(n+a)^b$ is $O(n^b)$

$(n+a)^b \geq (n/2)^b \quad \text{for} \quad n \geq 2|a| \quad \text{(even if} \ a < 0)\n
= 2^{-b} n^b$

$= c’n \quad \text{for} \quad c’ = 2^{-b}$

so $(n+a)^b$ is $\Omega(n^b)$
Example: \( \sum_{1 \leq i \leq n} i = \Theta(n^2) \)

Proof:

(a) An upper bound: each term is \( \leq \) the max term
\[
\sum_{1 \leq i \leq n} i \leq \sum_{1 \leq i \leq n} n = n^2 = O(n^2)
\]

(b) A lower bound: each term is \( \geq \) the min term
\[
\sum_{1 \leq i \leq n} i \geq \sum_{1 \leq i \leq n} 1 = n = \Omega(n)
\]

This is valid, but a weak bound. Better: pick a large subset of large terms
\[
\sum_{1 \leq i \leq n} i \geq \sum_{n/2 \leq i \leq n} \frac{n}{2} \geq \left\lfloor \frac{n}{2} \right\rfloor^2 = \Omega(n^2)
\]
properties

Transitivity.
If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.
If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.
If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.

Additivity.
If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.
If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.
If $f = \Theta(h)$ and $g = O(h)$ then $f + g = \Theta(h)$.

Proofs are left as exercises.
For all \( r > 1 \) (no matter how small) and all \( d > 0 \), (no matter how large)
\[ n^d = O(r^n) \]

In short, every exponential grows faster than every polynomial!

(proof below)
Example: For any $a, b > 1$, $\log_a n$ is $\Theta(\log_b n)$

$log_a b = x$ means $a^x = b$

$a^{\log_a b} = b$

$(a^{\log_a b})^{\log_b n} = b^{\log_b n} = n$

$(\log_a b)(\log_b n) = \log_a n$

$c \log_b n = \log_a n$ for the constant $c = \log_a b$

So:

$log_b n = \Theta(\log_a n) = \Theta(\log n)$

Corollary: base of a log factor is usually irrelevant, asymptotically. E.g. “$O(n \log n)$” [but $n^{\log_2 8} \neq O(n^{\log_8 8})$]
Logarithms:
For all $x > 0$, *(no matter how small)* $\log n = O(n^x)$

*log grows slower than every polynomial*
f(n) is $o(g(n))$ iff $\lim_{n \to \infty} f(n)/g(n)=0$
that is $g(n)$ dominates $f(n)$

If $a \leq b$ then $n^a$ is $O(n^b)$

If $a < b$ then $n^a$ is $o(n^b)$

$f(n) = O(g(n))$ vs $f(n) = o(g(n))$ are analogs to $\leq$ vs $<$

Note:
if $f(n)$ is $\Theta(g(n))$ then it cannot be $o(g(n))$
\[ n^2 = o(n^3) \text{ [Use algebra]}: \]

\[ \lim_{n \to \infty} \frac{n^2}{n^3} = \lim_{n \to \infty} \frac{1}{n} = 0 \]

\[ n^3 = o(e^n) \text{ [Use L’Hospital’s rule 3 times]}: \]

\[ \lim_{n \to \infty} \frac{n^3}{e^n} = \lim_{n \to \infty} \frac{3n^2}{e^n} = \lim_{n \to \infty} \frac{6n}{e^n} = \lim_{n \to \infty} \frac{6}{e^n} = 0 \]
For all \( r > 1 \) (no matter how small) and all \( d > 0 \), (no matter how large)
\[
n^d = \Theta(r^n) \\
n^d = o(r^n), \text{ even}
\]
Exercise: prove this, using tricks from previous slide

In short, every exponential grows faster than every polynomial!
Given two functions \( f(n) \) and \( g(n) \), if

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 
  c & \text{for some constant } c > 0 \\
  0 & 
\end{cases}
\]

then

\[
\begin{cases} 
  f(n) = \Theta(g(n)) \\
  f(n) = o(g(n)) \ [ \Rightarrow O(g(n))] 
\end{cases}
\]

respectively.

Inconclusive if the limit doesn’t exist. E.g., no limit for \( f/g \) at right, but \( g(n) \leq f(n) = O(f(n)) \)
big-theta, etc. are not always “nice”

\[ f(n) = \begin{cases} 
  n^2, & n \text{ even} \\
  n, & n \text{ odd} 
\end{cases} \]

\[ f(n) \neq \Theta(n^a) \text{ for any } a. \]

Fortunately, such nasty cases are rare

\[ n \log n \neq \Theta(n^a) \text{ for any } a, \text{ either, but at least it’s simpler.} \]
“Theorem”: $\sum_{1 \leq i \leq n} i = O(n)$

“Proof:” (by induction on $n$)

basis: $\sum_{1 \leq i \leq 1} i = 1 = O(1)$

induction step:

$$\sum_{1 \leq i \leq n} i = (\sum_{1 \leq i \leq n-1} i) + n$$

$$= O(n-1) + n \quad \text{(by ind. hyp.)}$$

$$= O(n)$$

Q. Where’s the flaw??

A. Never use “big-O” like this in an induction; instead, explicitly show the implicit constant “c”; in the above “proof,” you’ll see “c” become “c+1”…
Polynomial Time
the complexity class P: polynomial time

P: The set of problems solvable by algorithms with running time $O(n^d)$ for some constant $d$

(d is a constant independent of the input size $n$)

*Nice scaling property:* there is a constant $c$ s.t. *doubling* $n$, time increases only by a factor of $c$.

(E.g., $c \sim 2^d$)

Contrast with exponential: For any constant $c$, there is a $d$ such that $n \rightarrow n+d$ increases time by a factor of more than $c$.

(E.g., $c = 100$ and $d = 7$ for $2^n$ vs $2^{n+7}$)
polynomial vs exponential growth

\[ 2^{2n} \]

\[ 2^{n/10} \]

\[ 1000n^2 \]
why it matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds $10^{25}$ years, we simply record the algorithm as taking a very long time.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n$</th>
<th>$n \log_2 n$</th>
<th>$n^2$</th>
<th>$n^3$</th>
<th>$1.5^n$</th>
<th>$2^n$</th>
<th>$n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 10$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>4 sec</td>
</tr>
<tr>
<td>$n = 30$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>18 min</td>
<td>$10^{25}$ years</td>
</tr>
<tr>
<td>$n = 50$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>11 min</td>
<td>36 years</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 100$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>1 sec</td>
<td>12,892 years</td>
<td>$10^{17}$ years</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 1,000$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>1 sec</td>
<td>18 min</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 10,000$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>2 min</td>
<td>12 days</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 100,000$</td>
<td>$&lt; 1$ sec</td>
<td>2 sec</td>
<td>3 hours</td>
<td>32 years</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 1,000,000$</td>
<td>1 sec</td>
<td>20 sec</td>
<td>12 days</td>
<td>31,710 years</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
</tbody>
</table>

not only get very big, but do so abruptly, which likely yields erratic performance on small instances
Next year’s computer will be 2x faster. If I can solve problem of size $n_0$ today, how large a problem can I solve in the same time next year?

<table>
<thead>
<tr>
<th>Complexity</th>
<th>Size Increase</th>
<th>E.g. $T=10^{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(n)$</td>
<td>$n_0 \rightarrow 2n_0$</td>
<td>$10^{12} \rightarrow 2 \times 10^{12}$</td>
</tr>
<tr>
<td>$O(n^2)$</td>
<td>$n_0 \rightarrow \sqrt{2} \ n_0$</td>
<td>$10^6 \rightarrow 1.4 \times 10^6$</td>
</tr>
<tr>
<td>$O(n^3)$</td>
<td>$n_0 \rightarrow 3\sqrt{2} \ n_0$</td>
<td>$10^4 \rightarrow 1.25 \times 10^4$</td>
</tr>
<tr>
<td>$2^{n/10}$</td>
<td>$n_0 \rightarrow n_0+10$</td>
<td>$400 \rightarrow 410$</td>
</tr>
<tr>
<td>$2^n$</td>
<td>$n_0 \rightarrow n_0+1$</td>
<td>$40 \rightarrow 41$</td>
</tr>
</tbody>
</table>
why “polynomial”?

Point is not that $n^{2000}$ is a nice time bound, or that the differences among $n$ and $2n$ and $n^2$ are negligible.

Rather, simple theoretical tools may not easily capture such differences, whereas exponentials are qualitatively different from polynomials, so more amenable to theoretical analysis.

“My problem is in P” is a starting point for a more detailed analysis

“My problem is not in P” may suggest that you need to shift to a more tractable variant, or otherwise readjust expectations
Summary
Typical initial goal for algorithm analysis is to find an asymptotic upper bound on worst case running time as a function of problem size. This is rarely the last word, but often helps separate good algorithms from blatantly poor ones - concentrate on the good ones!