CSE 421
Algorithms

Huffman Codes:
An Optimal Data Compression Method
Compression Example

100k file, 6 letter alphabet:

File Size:

ASCII, 8 bits/char: 800kbits
$2^3 > 6$; 3 bits/char: 300kbits

Why?

Storage, transmission vs 5 Ghz cpu
Compression Example

100k file, 6 letter alphabet:

File Size:

- ASCII, 8 bits/char: 800kbits
- $2^3 > 6$; 3 bits/char: 300kbits
- better: $2.52 \text{ bits/char} = 74\% \times 2 + 26\% \times 4$: 252kbits
- Optimal?

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<thead>
<tr>
<th></th>
<th>%</th>
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</thead>
<tbody>
<tr>
<td>a</td>
<td>45</td>
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<tr>
<td>b</td>
<td>13</td>
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<tr>
<td>c</td>
<td>12</td>
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<td>d</td>
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<td>e</td>
<td>9</td>
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<tr>
<td>f</td>
<td>5</td>
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E.g.: Why not:
- a 00 00
- b 01 01
- d 10 10
- c 1100 110
- e 1101 1101
- f 1110 1110

1101110 = cf or ec?
Data Compression

Binary character code ("code")
- each k-bit source string maps to unique code word
  (e.g. k=8)
- “compression” alg: concatenate code words for successive k-bit “characters” of source

Fixed/variable length codes
- all code words equal length?

Prefix codes
- no code word is prefix of another (unique decoding)
Prefix Codes = Trees

<table>
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<th>Symbol</th>
<th>Frequency</th>
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<td>45%</td>
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Binary codes for messages:

- f: 1 0 1 0 0 0 0 0 1
- a: 1 1 0 0 0 0 1 0 1
Greedy Idea #1

Put most frequent under root, then recurse ...

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Greedy Idea #1

Top down: Put *most* frequent under root, then recurse

Too greedy: unbalanced tree

\[0.45 \times 1 + 0.16 \times 2 + 0.13 \times 3 \ldots = 2.34\]

not too bad, but imagine if all freqs were \(\sim 1/6\):

\[
\frac{(1+2+3+4+5+5)}{6} = 3.33
\]
Greedy Idea #2

Top down: Divide letters into 2 groups, with ~50% weight in each; recurse (Shannon-Fano code)

Again, not terrible
2*.5+3*.5 = 2.5

But this tree can easily be improved! (How?)
Greedy idea #3

Bottom up: Group least frequent letters near bottom

<table>
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Diagram:

- 100
- 25
  - c:12
  - b:13
- 14
  - f:5
  - e:9
.45*1 + .41*3 + .14*4 = 2.24 bits per char
Huffman’s Algorithm (1952)

Algorithm:

insert node for each letter into priority queue by freq
while queue length > 1 do
    remove smallest 2; call them x, y
    make new node z from them, with f(z) = f(x) + f(y)
    insert z into queue

Analysis: $O(n)$ heap ops: $O(n \log n)$

Goal: Minimize $B(T) = \sum_{c \in C} \text{freq}(c) \times \text{depth}(c)$

Correctness: ???
Correctness Strategy

Optimal solution may not be unique, so cannot prove that greedy gives the only possible answer.

Instead, show that greedy’s solution is as good as any.

How: an exchange argument
Claim: If we flip an inversion, cost never increases.

Why? All other things being equal, better to give more frequent letter the shorter code.

I.e., non-negative cost savings.

Defn: A pair of leaves is an inversion if

\[ \text{depth}(x) \geq \text{depth}(y) \]

and

\[ \text{freq}(x) \geq \text{freq}(y) \]

Claim: If we flip an inversion, cost never increases.

Why? All other things being equal, better to give more frequent letter the shorter code.

\[
\begin{align*}
\text{before:} & \\
(d(x)f(x) + d(y)f(y)) - (d(x)f(y) + d(y)f(x)) = \\
\text{after:} & \\
(d(x) - d(y)) * (f(x) - f(y)) \geq 0
\end{align*}
\]

I.e., non-negative cost savings.
Lemma 1: “Greedy Choice Property”

The 2 least frequent letters might as well be siblings at deepest level

Let a be least freq, b 2nd

Let u, v be siblings at max depth, f(u) ≤ f(v)
(why must they exist?)

Then (a,u) and (b,v) are inversions. Swap them.
Lemma 2

Let \((C, f)\) be a problem instance: \(C\) an \(n\)-letter alphabet with letter frequencies \(f(c)\) for \(c\) in \(C\).

For any \(x, y\) in \(C\), \(z\) not in \(C\), let \(C'\) be the \((n-1)\) letter alphabet \(C - \{x,y\} \cup \{z\}\) and for all \(c\) in \(C'\) define

\[
  f'(c) = \begin{cases} 
  f(c), & \text{if } c \neq x,y,z \\
  f(x) + f(y), & \text{if } c = z 
  \end{cases}
\]

Let \(T'\) be an optimal tree for \((C', f')\).

Then

\[
  T = T'
\]

is optimal for \((C, f)\) among all trees having \(x, y\) as siblings.
Proof:

\[ B(T) = \sum_{c \in C} d_T(c) \cdot f(c) \]

\[ B(T) - B(T') = d_T(x) \cdot (f(x) + f(y)) - d_{T'}(z) \cdot f'(z) \]

\[ = (d_{T'}(z) + 1) \cdot f'(z) - d_{T'}(z) \cdot f'(z) \]

\[ = f'(z) \]

Suppose \( \hat{T} \) (having \( x \) & \( y \) as siblings) is better than \( T \), i.e.

\[ B(\hat{T}) < B(T) \]. Collapse \( x \) & \( y \) to \( z \), forming \( \hat{T}' \); as above:

\[ B(\hat{T}) - B(\hat{T}') = f'(z) \]

Then:

\[ B(\hat{T}') = B(\hat{T}) - f'(z) < B(T) - f'(z) = B(T') \]

Contradicting optimality of \( T' \)
Theorem: Huffman gives optimal codes

Proof: induction on $|C|$

Basis: $n=1,2$ – immediate

Induction: $n>2$

Let $x,y$ be least frequent

Form $C', f', \& z$, as above

By induction, $T'$ is opt for $(C', f')$

By lemma 2, $T' \rightarrow T$ is opt for $(C, f)$ among trees with $x,y$ as siblings

By lemma 1, some opt tree has $x,y$ as siblings

Therefore, $T$ is optimal.
Data Compression

Huffman is optimal.

BUT still might do better!

Huffman encodes fixed length blocks. What if we vary them?

Huffman uses one encoding throughout a file. What if characteristics change?

What if data has structure? E.g. raster images, video,…

Huffman is lossless. Necessary?

LZW, MPEG, …
David A. Huffman, 1925-1999