Chapter 8
NP and Computational Intractability

Algorithm Design Patterns and Anti-Patterns

Algorithm design patterns.
- Greedy.
- Divide-and-conquer.
- Dynamic programming.
- Duality.
- Reductions.
- Local search.
- Randomization.

Algorithm design anti-patterns.
- NP-completeness.
- PSPACE-completeness.
- Undecidability.

8.1 Polynomial-Time Reductions

Classify Problems According to Computational Requirements

Q. Which problems will we be able to solve in practice?

Those with polynomial-time algorithms.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Polynomial-time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shortest path</td>
<td>Yes</td>
</tr>
<tr>
<td>Matching</td>
<td>Yes</td>
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<tr>
<td>Min cut</td>
<td>Yes</td>
</tr>
<tr>
<td>2-SAT</td>
<td>Yes</td>
</tr>
<tr>
<td>Planar 4-color</td>
<td>No</td>
</tr>
<tr>
<td>Bipartite vertex cover</td>
<td>No</td>
</tr>
<tr>
<td>Primality testing</td>
<td>No</td>
</tr>
<tr>
<td>Longest path</td>
<td>Probably no</td>
</tr>
<tr>
<td>3D-matching</td>
<td>Probably no</td>
</tr>
<tr>
<td>Max cut</td>
<td>No</td>
</tr>
<tr>
<td>3-SAT</td>
<td>No</td>
</tr>
<tr>
<td>Planar 3-color</td>
<td>No</td>
</tr>
<tr>
<td>Vertex cover</td>
<td>No</td>
</tr>
<tr>
<td>Factoring</td>
<td>No</td>
</tr>
</tbody>
</table>
Classify Problems

Desiderata. Classify problems according to those that can be solved in polynomial-time and those that cannot.

Provably requires exponential-time.
- Given a Turing machine, does it halt in at most k steps?
- Given a board position in an n-by-n generalization of chess, can black guarantee a win?

Frustrating news. Huge number of fundamental problems have defied classification for decades.

This chapter. Show that these fundamental problems are "computationally equivalent" and appear to be different manifestations of one really hard problem.

Polynomial-Time Reduction

Desiderata'. Suppose we could solve X in polynomial-time. What else could we solve in polynomial time?

Reduction. Problem X polynomial reduces to problem Y if arbitrary instances of problem X can be solved using:
- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

Notation. $X \leq_P Y$.

Remarks.
- We pay for time to write down instances sent to black box $\Rightarrow$ instances of Y must be of polynomial size.
- Note: Cook reducibility.

Reduction By Simple Equivalence

Purpose. Classify problems according to relative difficulty.

Design algorithms. If $X \leq_P Y$ and Y can be solved in polynomial-time, then X can also be solved in polynomial time.

Establish intractability. If $X \leq_P Y$ and X cannot be solved in polynomial-time, then Y cannot be solved in polynomial-time.

Establish equivalence. If $X \leq_P Y$ and $Y \leq_P X$, we use notation $X \equiv_P Y$.

Basic reduction strategies.
- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.
Independent Set

**INDEPENDENT SET**: Given a graph $G = (V, E)$ and an integer $k$, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in $S$?

**Ex.** Is there an independent set of size $\geq 6$? Yes.
**Ex.** Is there an independent set of size $\geq 7$? No.

Vertex Cover

**VERTEX COVER**: Given a graph $G = (V, E)$ and an integer $k$, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in $S$?

**Ex.** Is there a vertex cover of size $\leq 4$? Yes.
**Ex.** Is there a vertex cover of size $\leq 3$? No.

**Claim.** $\text{VERTEX-COVER} \equiv \text{INDEPENDENT-SET}$.

**Pf.** We show $S$ is an independent set iff $V - S$ is a vertex cover.

$\Rightarrow$
Let $S$ be any independent set.
Consider an arbitrary edge $(u, v)$.
$S$ independent $\Rightarrow u \notin S$ or $v \notin S$ $\Rightarrow u \in V - S$ or $v \in V - S$.
Thus, $V - S$ covers $(u, v)$.

$\Leftarrow$
Let $V - S$ be any vertex cover.
Consider two nodes $u \in S$ and $v \in S$.
Observe that $(u, v) \notin E$ since $V - S$ is a vertex cover.
Thus, no two nodes in $S$ are joined by an edge $\Rightarrow S$ independent set.
**Reduction from Special Case to General Case**

Basic reduction strategies.
- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

**Set Cover**

**Set Cover**: Given a set $U$ of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of $U$, and an integer $k$, does there exist a collection of $\leq k$ of these sets whose union is equal to $U$?

Sample application.
- $m$ available pieces of software.
- $U$ of $n$ capabilities that we would like our system to have.
- The $i$th piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all $n$ capabilities using fewest pieces of software.

**Ex:**

Given $U = \{1, 2, 3, 4, 5, 6, 7\}$, $k = 2$:
- $S_1 = \{3, 7\}$
- $S_2 = \{2, 4\}$
- $S_3 = \{3, 4, 5, 6\}$
- $S_4 = \{5\}$
- $S_5 = \{1\}$
- $S_6 = \{1, 2, 6, 7\}$

**Vertex Cover Reduces to Set Cover**

**Claim.** VERTEX-COVER $\leq_P$ SET-COVER.

**Pf.** Given a VERTEX-COVER instance $G = (V, E)$, $k$, we construct a set cover instance whose size equals the size of the vertex cover instance.

**Construction.**
- Create SET-COVER instance:
  - $k = 2$, $U = E$, $S_v = \{e \in E : e$ incident to $v\}$
  - Set-cover of size $\leq k$ iff vertex cover of size $\leq k$.

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**Polynomial-Time Reduction**

**Basic strategies.**
- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.
8.2 Reductions via "Gadgets"

Basic reduction strategies:
- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction via "gadgets."

3 Satisfiability Reduces to Independent Set

Claim. 3-SAT ≤ P INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k if Φ is satisfiable.

Construction.
- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

\[ \Phi = (x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor \overline{x_3}) \land (\overline{x_1} \lor x_2 \lor x_4) \]

k = 3

Satisfiability

Literal: A Boolean variable or its negation.
- \( x_i \) or \( \overline{x_i} \)

Clause: A disjunction of literals.
- \( C_j = x_i \lor \overline{x_i} \lor x_j \)

Conjunctive normal form: A propositional formula \( \Phi \) that is the conjunction of clauses.
- \( \Phi = C_1 \land C_2 \land C_3 \land C_4 \)

SAT: Given CNF formula \( \Phi \), does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

Ex: \( \Phi = (x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_4) \land (x_2 \lor x_3 \lor x_4) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3}) \)

Yes: \( x_1 = \text{true}, x_2 = \text{false} \).

3 Satisfiability Reduces to Independent Set

Claim. G contains independent set of size \( k = |\Phi| \) iff \( \Phi \) is satisfiable.

Pf. \( \Rightarrow \) Let \( S \) be independent set of size \( k \).
- \( S \) must contain exactly one vertex in each triangle.
- Set these literals to true, \( \cdots \) and any other variables in a consistent way.
- Truth assignment is consistent and all clauses are satisfied.

Pf. \( \Leftarrow \) Given satisfying assignment, select one true literal from each triangle. This is an independent set of size \( k \).
**Review**

**Basic reduction strategies**
- Simple equivalence: INDEPENDENT-SET $\leq_p$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\leq_p$ SET-COVER.
- Encoding with gadgets: 3-SAT $\leq_p$ INDEPENDENT-SET.

**Transitivity.** If $X \leq_p Y$ and $Y \leq_p Z$, then $X \leq_p Z$.

**Pf idea.** Compose the two algorithms.

**Ex:** 3-SAT $\leq_p$ INDEPENDENT-SET $\leq_p$ VERTEX-COVER $\leq_p$ SET-COVER.

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**Self-Reducibility**

**Decision problem.** Does there exist a vertex cover of size $\leq k$?

**Search problem.** Find vertex cover of minimum cardinality.

**Self-reducibility.** Search problem $\leq_p$ decision version.

- Applies to all (NP-complete) problems in this chapter.
- Justifies our focus on decision problems.

**Ex:** to find min cardinality vertex cover.

- (Binary) search for cardinality $k^*$ of min vertex cover.
  - Find a vertex $v$ such that $G - \{v\}$ has a vertex cover of size $\leq k^* - 1$.
    - any vertex in any min vertex cover will have this property
    - Include $v$ in the vertex cover.
    - Recursively find a min vertex cover in $G - \{v\}$.
  - delete $v$ and all incident edges

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**Reduction Practice 1**

**Clique Problem:** Given an undirected graph $G = (V,E)$ and number $k$, does there exist a clique of size $k$ in $G$? A clique is a set $S \subseteq V$ in which there is an edge between any two vertices in $S$.

Show that Independent Set is polynomial time reducible to Clique.

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**Reduction Practice 2**

**k-coloring problem:** Given an undirected graph $G = (V,E)$ does there exist a coloring of the graph with $k$ colors. A coloring of a graph with $k$ colors is a coloring of the vertices in $k$ colors so that no two adjacent vertices have the same color.

Show that 3-coloring is polynomial time reducible to $k$-coloring for any $k > 3$. 