Network Flow

Outline

• Network flow definitions
• Flow examples
• Augmenting Paths
• Residual Graph
• Ford Fulkerson Algorithm
• Cuts
• Maxflow-MinCut Theorem

Flow Example

Flow assignment and the residual graph

Network Flow Definitions

• Capacity
• Source, Sink
• Capacity Condition
• Conservation Condition
• Value of a flow
Network Flow Definitions

- **Flowgraph**: Directed graph with distinguished vertices $s$ (source) and $t$ (sink)
- Capacities on the edges, $c(e) \geq 0$
- Problem, assign flows $f(e)$ to the edges such that:
  - $0 \leq f(e) \leq c(e)$
  - Flow is conserved at vertices other than $s$ and $t$
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is as large as possible

**Flow Example**

```
Construct a maximum flow and indicate the flow value
```

Find a maximum flow

```
Value of flow:
```

Augmenting Path Algorithm

- Augmenting path
  - Vertices $v_1, v_2, \ldots, v_k$
    - $v_1 = s$, $v_k = t$
  - Possible to add $b$ units of flow between $v_j$ and $v_{j+1}$ for $j = 1 \ldots k-1$

Find two augmenting paths
Residual Graph

- Flow graph showing the remaining capacity
- Flow graph $G$, Residual Graph $G_R$
  - $G$: edge $e$ from $u$ to $v$ with capacity $c$ and flow $f$
  - $G_R$: edge $e'$ from $u$ to $v$ with capacity $c - f$
  - $G_R$: edge $e''$ from $v$ to $u$ with capacity $f$

Build the residual graph

Augmenting Path Lemma

- Let $P = v_1, v_2, ..., v_k$ be a path from $s$ to $t$ with minimum capacity $b$ in the residual graph.
- $b$ units of flow can be added along the path $P$ in the flow graph.

Proof

- Add $b$ units of flow along the path $P$
- What do we need to verify to show we have a valid flow after we do this?
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Ford-Fulkerson Algorithm (1956)

while not done
  Construct residual graph $G_R$
  Find an s-t path $P$ in $G_R$ with capacity $b > 0$
  Add $b$ units along in $G$

If the sum of the capacities of edges leaving $S$ is at most $C$, then the algorithm takes at most $C$ iterations.