Greedy Algorithms

• Solve problems with the simplest possible algorithm
• The hard part: showing that something simple actually works
• Today’s problems (Sections 4.2, 4.3)
  – Homework Scheduling
  – Optimal Caching

Homework Scheduling

• Tasks to perform
• Deadlines on the tasks
• Freedom to schedule tasks in any order

• Can I get all my work turned in on time?
• If I can’t get everything, I want to minimize the maximum lateness

Scheduling tasks

• Each task has a length $t_i$ and a deadline $d_i$
• All tasks are available at the start
• One task may be worked on at a time
• All tasks must be completed

• Goal: minimize maximum lateness
  – Lateness = $f_i - d_i$ if $f_i \geq d_i$

Example

<table>
<thead>
<tr>
<th>Time</th>
<th>Deadline</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
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<tr>
<td>3</td>
<td>4</td>
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Lateness 1

<table>
<thead>
<tr>
<th>Time</th>
<th>Deadline</th>
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<tbody>
<tr>
<td>2</td>
<td>3</td>
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</table>

Lateness 3

Determine the minimum lateness

<table>
<thead>
<tr>
<th>Time</th>
<th>Deadline</th>
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<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
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<tr>
<td>4</td>
<td>5</td>
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<tr>
<td>5</td>
<td>12</td>
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</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Deadline</th>
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<tbody>
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<td>2</td>
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<td>4</td>
<td>5</td>
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<tr>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>
Greedy Algorithm

- Earliest deadline first
- Order jobs by deadline
- This algorithm is optimal

Analysis

- Suppose the jobs are ordered by deadlines, \( d_1 \leq d_2 \leq \ldots \leq d_n \)
- A schedule has an inversion if job \( j \) is scheduled before \( i \) where \( j > i \)
- The schedule \( A \) computed by the greedy algorithm has no inversions.
- Let \( O \) be the optimal schedule, we want to show that \( A \) has the same maximum lateness as \( O \)

List the inversions

<table>
<thead>
<tr>
<th></th>
<th>Time</th>
<th>Deadline</th>
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</thead>
<tbody>
<tr>
<td>a₁</td>
<td>3</td>
<td>4</td>
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<tr>
<td>a₂</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>a₃</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>a₄</td>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>

Lemma: There is an optimal schedule with no idle time

- It doesn’t hurt to start your homework early!
- Note on proof techniques
  - This type of can be important for keeping proofs clean
  - It allows us to make a simplifying assumption for the remainder of the proof

Lemma

- If there is an inversion \( i, j \), there is a pair of adjacent jobs \( i', j' \) which form an inversion

Interchange argument

- Suppose there is a pair of jobs \( i \) and \( j \), with \( d_i \leq d_j \), and \( j \) scheduled immediately before \( i \). Interchanging \( i \) and \( j \) does not increase the maximum lateness.
**Proof by Bubble Sort**

Determine maximum lateness

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**Real Proof**

- There is an optimal schedule with no inversions and no idle time.
- Let O be an optimal schedule k inversions, we construct a new optimal schedule with k-1 inversions.
- Repeat until we have an optimal schedule with 0 inversions.
- This is the solution found by the earliest deadline first algorithm.

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**Result**

- Earliest Deadline First algorithm constructs a schedule that minimizes the maximum lateness.

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**Homework Scheduling**

- How is the model unrealistic?

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**Extensions**

- What if the objective is to minimize the sum of the lateness?
  - EDF does not seem to work.
- If the tasks have release times and deadlines, and are non-preemptable, the problem is NP-complete.
- What about the case with release times and deadlines where tasks are preemptable?

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**Optimal Caching**

- Caching problem:
  - Maintain collection of items in local memory.
  - Minimize number of items fetched.
Caching example

A, B, C, D, A, E, B, A, D, A, C, B, D, A

Optimal Caching

- If you know the sequence of requests, what is the optimal replacement pattern?
- Note – it is rare to know what the requests are in advance – but we still might want to do this:
  - Some specific applications, the sequence is known
  - Competitive analysis, compare performance on an online algorithm with an optimal offline algorithm

Farthest in the future algorithm

- Discard element used farthest in the future

A, B, C, A, C, D, C, B, C, A, D

Correctness Proof

- Sketch
- Start with Optimal Solution O
- Convert to Farthest in the Future Solution F-F
- Look at the first place where they differ
- Convert O to evict F-F element
  - There are some technicalities here to ensure the caches have the same configuration . . .

Subsequence Testing

- Is $a_1a_2...a_m$ a subsequence of $b_1b_2...b_n$?
  - e.g. S,A,G,E is a subsequence of S,T,U,A,R,T,R,E,G,E,S

Greedy Algorithm for Subsequence Testing