CSE 421: Intro Algorithms
Summer 2007
Graphs and Graph Algorithms
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Objects & Relationships
The Kevin Bacon Game:
Actors
Two are related if they’ve been in a movie together
Exam Scheduling:
Classes
Two are related if they have students in common
Traveling Salesperson Problem:
Cities
Two are related if can travel directly between them

Graphs
An extremely important formalism for representing (binary) relationships
Objects: “vertices”, aka “nodes”
Relationships between pairs: “edges”, aka “arcs”
Formally, a graph $G = (V, E)$ is a pair of sets, $V$ the vertices and $E$ the edges
Undirected Graph $G = (V,E)$

"self-loop"

"multi-edge"
Undirected Graph $G = (V,E)$

Directed Graph $G = (V,E)$

Graphs don’t live in Flatland

Geometrical drawing is mentally convenient, but mathematically irrelevant: 4 drawings, 1 graph.
Directed Graph $G = (V,E)$

What are the vertices?
Explicitly list them:
{“A”, “7”, “3”, “4”}

What are the edges?
Either, set of edges
{⟨A,3⟩, ⟨7,4⟩, ⟨4,3⟩, ⟨4,A⟩}
Or, (symmetric) adjacency matrix:

\[
\begin{array}{cccc}
A & 7 & 3 & 4 \\
\hline
A & 0 & 1 & 1 \\
7 & 0 & 0 & 1 \\
3 & 1 & 0 & 0 \\
4 & 1 & 1 & 1 \\
\end{array}
\]
Specifying directed graphs as input

What are the vertices?
Explicitly list them:
{“A”, “7”, “3”, “4”}

What are the edges?
Either, set of directed edges:
{(A,4), (4,7), (4,3), (4,A), (A,3)}
Or, (nonsymmetric) adjacency matrix:

```
A  7  3  4
A  0  0  1  1
7  0  0  0  0
3  0  0  0  0
4  1  1  1  0
```

More Cool Graph Lingo

A graph is called **sparse** if \( m \ll n^2 \), otherwise it is **dense**.

Boundary is somewhat fuzzy; \( O(n) \) edges is certainly sparse, \( \Omega(n^2) \) edges is dense.

Sparse graphs are common in practice
E.g., all planar graphs are sparse (\( m \leq 3n-6 \), for \( n \geq 3 \))

Q: which is a better run time, \( O(n+m) \) or \( O(n^2) \)?

A: \( O(n+m) = O(n^2) \), but \( n+m \) usually way better!

# Vertices vs # Edges

Let \( G \) be an undirected graph with \( n \) vertices and \( m \) edges. How are \( n \) and \( m \) related?

Since every edge connects two different vertices (no loops), and no two edges connect the same two vertices (no multi-edges),
it must be true that:

\[
0 \leq m \leq \frac{n(n-1)}{2} = O(n^2)
\]

Representing Graph \( G = (V,E) \)

**Vertex set** \( V = \{v_1, \ldots, v_n\} \)

**Adjacency Matrix** \( A \)

\[
A[i,j] = 1 \text{ iff } (v_i,v_j) \in E
\]

Space is \( n^2 \) bits

**Advantages:**
\( O(1) \) test for presence or absence of edges.

**Disadvantages:** inefficient for sparse graphs, both in storage and access
**Representing Graph** \( G=(V,E) \)

- **n** vertices, **m** edges

**Adjacency List:**

\[ O(n+m) \text{ words} \]

**Advantages:**
- Compact for sparse graphs
- Easily see all edges

**Disadvantages:**
- More complex data structure
- No \( O(1) \) edge test

**Back- and cross pointers more work to build, but allow easier traversal and deletion of edges, if needed, (don’t bother if not)**

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**Graph Traversal**

Learn the basic structure of a graph

“Walk,” via edges, from a fixed starting vertex \( s \) to all vertices reachable from \( s \)

Being **orderly** helps. Two common ways:

- Breadth-First Search
- Depth-First Search

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**Breadth-First Search**

Completely explore the vertices in order of their distance from \( s \)

Naturally implemented using a queue
Breadth-First Search

Idea: Explore from s in all possible directions, layer by layer.

BFS algorithm.

- $L_0 = \{ s \}$.
- $L_1$ = all neighbors of $L_0$.
- $L_2$ = all nodes not in $L_0$ or $L_1$, and having an edge to a node in $L_1$.
- $L_{i+1}$ = all nodes not in earlier layers, and having an edge to a node in $L_i$.

Theorem. For each $i$, $L_i$ consists of all nodes at distance (i.e., min path length) exactly $i$ from $s$.

Cor: There is a path from $s$ to $t$ iff $t$ appears in some layer.

Graph Traversal: Implementation

Learn the basic structure of a graph

“Walk,” via edges, from a fixed starting vertex $s$ to all vertices reachable from $s$

Three states of vertices

- undiscovered
- discovered
- fully-explored

BFS(s) Implementation

Global initialization: mark all vertices "undiscovered"

BFS(s)

- mark s "discovered"
- queue = { s }
- while queue not empty
  - u = remove_first(queue)
  - for each edge {u,x}
    - if (x is undiscovered)
      - mark x discovered
      - append x on queue
    - mark u fully explored

Exercise: modify code to number vertices & compute level numbers
BFS(v)

Queue: 2 3

BFS(v)

Queue: 3 4

BFS(v)

Queue: 4 5 6 7

BFS(v)

Queue: 5 6 7 8 9
BFS(v)

Queue: 8 9 10 11

BFS(v)

Queue: 10 11 12 13

BFS(v)

Queue: 12 13 14

BFS(s) Implementation

Global initialization: mark all vertices "undiscovered"

BFS(s)
mark s "discovered"
queue = { s }
while queue not empty
u = remove_first(queue)
for each edge (u,x)
if (x is undiscovered)
mark x discovered
append x on queue
mark u fully explored

Exercise: modify code to number vertices & compute level numbers
**BFS analysis**

Each edge is explored once from each end-point.

Each vertex is discovered by following a different edge.

Total cost $O(m)$, $m = \#$ of edges.

**Properties of (Undirected) BFS($v$)**

BFS(v) visits x if and only if there is a path in G from v to x.

Edges into then-undiscovered vertices define a tree – the "breadth first spanning tree" of G.

Level i in this tree are exactly those vertices $u$ such that the shortest path (in G, not just the tree) from the root v is of length i.

All non-tree edges join vertices on the same or adjacent levels.

**BFS Application: Shortest Paths**

Tree (solid edges) gives shortest paths from start vertex.

Can label by distances from start.

All edges connect same/adjacent levels.
Why fuss about trees?

Trees are simpler than graphs

Ditto for algorithms on trees vs algs on graphs

So, this is often a good way to approach a graph problem: find a "nice" tree in the graph, i.e., one such that non-tree edges have some simplifying structure

E.g., BFS finds a tree s.t. level-jumps are minimized

DFS (next) finds a different tree, but it also has interesting structure…

Graph Search Application: Connected Components

Want to answer questions of the form: given vertices u and v, is there a path from u to v?

Idea: create array A such that


Q: Why not create 2-d array Path[u,v]?
**Graph Search Application:**

**Connected Components**

Initial state: all $v$ undiscovered.

For $v = 1$ to $n$ do

if state($v$) != fully-explored then

BFS($v$): setting $A[u] \leftarrow v$ for each $u$ found

(and marking $u$ discovered/fully-explored)

endif

endfor

Total cost: $O(n+m)$

each edge is touched a constant number of times (twice)

works also with DFS

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**Bipartite Graphs**

**Definition.** An undirected graph $G = (V, E)$ is **bipartite (2-colorable)** if the nodes can be colored red or blue such that no edge has both ends the same color.

**Applications.**

Stable marriage: men = red, women = blue

Scheduling: machines = red, jobs = blue

An equivalent definition: $G$ is bipartite if you can partition the node set into 2 parts (say, blue/red or left/right) so that all edges join nodes in different parts/no edge has both ends in the same part.

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**3.4 Testing Bipartiteness**

**Testing Bipartiteness.** Given a graph $G$, is it bipartite?

Many graph problems become:

easier if the underlying graph is bipartite (matching)

tractable if the underlying graph is bipartite (independent set)

Before attempting to design an algorithm, we need to understand structure of bipartite graphs.
**An Obstruction to Bipartiteness**

Lemma. If a graph $G$ is bipartite, it cannot contain an odd length cycle.

**Pf.** Impossible to 2-color the odd cycle, let alone $G$.

**Bipartite Graphs**

Lemma. Let $G$ be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.

(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).

**Pf.**

(i) Suppose no edge joins two nodes in the same layer. By previous lemma, all edges join nodes on adjacent levels. Bipartition:
red = nodes on odd levels, blue = nodes on even levels.

(ii) Suppose $(x, y)$ is an edge & $x, y$ in same level $L_j$. Let $z$ = their lowest common ancestor in BFS tree. Let $L_i$ be level containing $z$. Consider cycle that takes edge from $x$ to $y$, then tree from $y$ to $z$, then tree from $z$ to $x$. Its length is $1 + (j-i) + (j-i)$, which is odd.
Obstruction to Bipartiteness

Cor: A graph $G$ is bipartite iff it contains no odd length cycle.

NB: the proof is algorithmic—it finds a coloring or odd cycle.

3.6 DAGs and Topological Ordering

Directed Acyclic Graphs

Def. A **DAG** is a directed acyclic graph, i.e., one that contains no directed cycles.

Ex. Precedence constraints: edge $(v_i, v_j)$ means $v_i$ must precede $v_j$.

Def. A **topological order** of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge $(v_i, v_j)$ we have $i < j$.

Precedence Constraints

Precedence constraints. Edge $(v_i, v_j)$ means task $v_i$ must occur before $v_j$.

Applications

Course prerequisite graph: course $v_i$ must be taken before $v_j$.

Compilation: must compile module $v_i$ before $v_j$.

Pipeline of computing jobs: output of job $v_i$ is part of input to job $v_j$.

Manufacturing or assembly: sand it before you paint it.

Spreadsheet evaluation order: if A7 is “=A6+A5+A4”, evaluate them 1st.

Directed Acyclic Graphs
Directed Acyclic Graphs

Lemma. If G has a topological order, then G is a DAG.

Pf. (by contradiction)

Suppose that G has a topological order $v_1, \ldots, v_n$ and that G also has a directed cycle C.
Let $v_i$ be the lowest-indexed node in C, and let $v_j$ be the node just before $v_i$; thus ($v_j, v_i$) is an edge.
By our choice of i, we have $i < j$.
On the other hand, since ($v_j, v_i$) is an edge and $v_1, \ldots, v_n$ is a topological order, we must have $j < i$, a contradiction. ▪

Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)

Suppose that G is a DAG and every node has at least one incoming edge. Let’s see what happens.
Pick any node $v$, and begin following edges backward from $v$. Since $v$ has at least one incoming edge ($u, v$) we can walk backward to $u$.
Then, since $u$ has at least one incoming edge ($x, u$), we can walk backward to $x$.
Repeat until we visit a node, say $w$, twice.
Let $C$ be the sequence of nodes encountered between successive visits to $w$. $C$ is a cycle.

Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)

Base case: true if $n = 1$.
Given DAG on $n > 1$ nodes, find a node $v$ with no incoming edges. $G - \{v\}$ is a DAG, since deleting $v$ cannot create cycles.
By inductive hypothesis, $G - \{v\}$ has a topological ordering.
Place $v$ first in topological ordering; then append nodes of $G - \{v\}$ in topological order. This is valid since $v$ has no incoming edges. ▪

To compute a topological ordering of G:
Find a node $v$ with no incoming edges and order it first.
Delete $v$ from G.
Recursively compute a topological ordering of $G - \{v\}$ and append this order after $v$. 
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2$

Topological order: $v_1, v_2, v_3$
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4$

Topological order: $v_1, v_2, v_3, v_4, v_5$

Topological order: $v_1, v_2, v_3, v_4, v_5, v_6$

Topological order: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$. 
Topological Sorting Algorithm

Maintain the following:
\[ \text{count}[w] = (\text{remaining}) \text{ number of incoming edges to node } w \]
\[ S = \text{set of (remaining) nodes with no incoming edges} \]

Initialization:
\[ \text{count}[w] = 0 \text{ for all } w \]
\[ \text{count}[w]++ \text{ for all edges } (v,w) \]
\[ S = S \cup \{w\} \text{ for all } w \text{ with } \text{count}[w]==0 \]

Main loop:
\[ \text{while } S \text{ not empty} \]
\[ \text{remove some } v \text{ from } S \]
\[ \text{make } v \text{ next in topo order} \]
\[ \text{for all edges from } v \text{ to some } w \]
\[ \text{decrement } \text{count}[w] \]
\[ \text{add } w \text{ to } S \text{ if } \text{count}[w] \text{ hits 0} \]

Correctness: clear, I hope

Time: \( O(m + n) \) (assuming edge-list representation of graph)

Depth-First Search

Follow the first path you find as far as you can
Back up to last unexplored edge when you reach a dead end, then go as far you can

Naturally implemented using recursive calls or a stack

DFS(v) – Recursive version

Global Initialization:
\[ \text{for all nodes } v, \text{ v.dfs\#} = -1 \text{ // mark v "undiscovered"} \]
\[ \text{dfscounter} = 0 \]

DFS(v)
\[ \text{v.dfs\#} = \text{dfscounter}++ \text{ // v "discovered", number it} \]
\[ \text{for each edge } (v,x) \]
\[ \text{if } (x.dfs\# = -1) \text{ // tree edge (x previously undiscovered)} \]
\[ \text{DFS(x)} \]
\[ \text{else} \ldots \text{ // code for back-, fwd-, parent,} \]
\[ \text{edges, if needed} \]
\[ \text{mark v "completed," if needed} \]

DFS(A)

Color code:
- undiscovered
- discovered
- fully-explored

Call Stack
(Edge list):
A (B,J)

Suppose edge lists at each vertex are sorted alphabetically
Suppose edge lists at each vertex are sorted alphabetically.

**Color code:**
- undiscovered
- discovered
- fully-explored

**Call Stack:**
- (Edge list)

1. **DFS(A)**
   - A,1
   - B,2
   - J
   - C
   - G
   - H
   - K
   - L
   - D
   - F
   - I
   - M
   - E

2. **DFS(A)**
   - A,1
   - B,2
   - J
   - C,3
   - G
   - H
   - K
   - L
   - D
   - F
   - I
   - M
   - E

3. **DFS(A)**
   - A,1
   - B,2
   - J
   - C,3
   - G
   - H
   - K
   - L
   - D,4
   - F
   - I
   - M
   - E

4. **DFS(A)**
   - A,1
   - B,2
   - J
   - C,3
   - G
   - H
   - K
   - L
   - D,4
   - F
   - I
   - M
   - E,5
Suppose edge lists at each vertex are sorted alphabetically.

Color code:
- Undiscovered
- Discovered
- Fully-explored

Call Stack:
- (Edge list)
- A (B, J)
- B (A, C, J)
- C (B, D, G, H)
- D (C, E, F)
- E (D, F)
- F (D, E, G)

Call Stack:
- (Edge list)
- A (B, J)
- B (A, C, J)
- C (B, D, G, H)
- D (C, E, F)
- E (D, F)
- F (D, E, G)
- G (C, F)
Suppose edge lists at each vertex are sorted alphabetically.

Color code:
- undiscovered
- discovered
- fully-explored

Call Stack:
- (Edge list)

DFS(A):

1. A, B, C, D, E
2. A, B, J
3. A, B, C, J
4. A, B, C, J
5. A, B, J
6. A, B, J
7. A, B, C, J
8. A, B, C, J
9. A, B, C, J
10. A, B, C, J
11. A, B, C, J
12. A, B, C, J
13. A, B, C, J
15. A, B, C, J
16. A, B, C, J
17. A, B, C, J
18. A, B, C, J
19. A, B, C, J
20. A, B, C, J
21. A, B, C, J
22. A, B, C, J
23. A, B, C, J
25. A, B, C, J
26. A, B, C, J
27. A, B, C, J
28. A, B, C, J
29. A, B, C, J
30. A, B, C, J
31. A, B, C, J
32. A, B, C, J
33. A, B, C, J
34. A, B, C, J
35. A, B, C, J
36. A, B, C, J
37. A, B, C, J
38. A, B, C, J
39. A, B, C, J
40. A, B, C, J
41. A, B, C, J
42. A, B, C, J
43. A, B, C, J
44. A, B, C, J
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73. A, B, C, J
74. A, B, C, J
75. A, B, C, J
76. A, B, C, J
77. A, B, C, J
78. A, B, C, J
79. A, B, C, J
80. A, B, C, J
81. A, B, C, J
82. A, B, C, J
83. A, B, C, J
84. A, B, C, J
85. A, B, C, J
86. A, B, C, J
Suppose edge lists at each vertex are sorted alphabetically.

**Color code:**
- undiscovered
- discovered
- fully-explored

**Call Stack:**
- (Edge list)

**DFS(A)**
Suppose edge lists at each vertex are sorted alphabetically.

**DFS(A)**

Call Stack:
- (Edge list)
- A (F,J)
- B (K,L)
- C (F,G,H)
- H (F,J)
- J (A,B,H,K,L)

**Color Code:**
- **undiscovered**
- **discovered**
- **fully-explored**

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Suppose edge lists at each vertex are sorted alphabetically.

Color code:
- **undiscovered**
- **discovered**
- **fully-explored**

Call Stack: (Edge list)
- A (F,J)
- B (G,F,J)
- C (G,F,J,H)
- H (G,F,J)
- J (G,F,J,K,L)
- K (G,F,J,L)
- L (G,F,J,M)

DFS(A)
Suppose edge lists at each vertex are sorted alphabetically

Color code:
- **undiscovered**
- **discovered**
- **fully-explored**

Call Stack:
(Edge list)

DFS(A)

A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
M,13
D,4
F,6
E,5

DFS(A)

A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
M,13
D,4
F,6
E,5

DFS(A)

A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
M,13
D,4
F,6
E,5

DFS(A)

A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
M,13
D,4
F,6
E,5

Suppose edge lists at each vertex are sorted alphabetically.

**Color code:**
- **undiscovered**
- **discovered**
- **fully-explored**

**Call Stack:**
- (Edge list) $A$ (F,J)
- (Edge list) $A$ (G,J)
- (Edge list) $A$ (F,J)

**Edge code:**
- **Tree edge**
- **Back edge**
Properties of (Undirected) DFS(v)

Like BFS(v):
DFS(v) visits \( x \) if and only if there is a path in \( G \) from \( v \) to \( x \) (through previously unvisited vertices)
Edges into then-undiscovered vertices define a tree – the "depth first spanning tree" of \( G \)

Unlike the BFS tree:
- the DF spanning tree isn't minimum depth
- its levels don't reflect min distance from the root
- non-tree edges never join vertices on the same or adjacent levels

BUT…

Non-tree edges

All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

No cross edges!
Why fuss about trees (again)?

As with BFS, DFS has found a tree in the graph s.t. non-tree edges are “simple”—only descendant/ancestor.

A simple problem on trees

*Given:* tree T, a value L(v) defined for every vertex v in T.

*Goal:* find M(v), the min value of L(v) anywhere in the subtree rooted at v (including v itself).

*How?* Depth first search, using:

\[
M(v) = \begin{cases} 
L(v) & \text{if v is a leaf} \\
\min(L(v), \min_{\text{a child of } v} M(w)) & \text{otherwise}
\end{cases}
\]

Application: Articulation Points

A node in an undirected graph is an **articulation point** iff removing it disconnects the graph.

articulation points represent vulnerabilities in a network — single points whose failure would split the network into 2 or more disconnected components.
Articulation Points

Articulation point iff its removal disconnects the graph

Simple Case: Artic. Pts in a tree
Leaves -- never articulation points
Internal nodes -- always articulation points
Root -- articulation point if and only if two or more children
Non-tree: extra edges remove some articulation points (which ones?)

Articulation Points from DFS
Root node is an articulation point iff it has more than one child
Leaf is never an articulation point
Non-leaf, non-root node $u$ is an articulation point

$\exists$ some child $y$ of $u$ s.t. no non-tree edge goes above $u$ from $y$ or below

If removal of $u$ does NOT separate $x$, there must be an exit from $x$'s subtree. How? Via back edge.
Articulation Points:
the "LOW" function

Definition: LOW(v) is the lowest dfs# of any vertex that is either in the dfs subtree rooted at v (including v itself) or connected to a vertex in that subtree by a back edge.

Key idea 1: if some child x of v has LOW(x) ≥ dfs#(v) then v is an articulation point (excl. root)

Key idea 2: LOW(v) = min ( {dfs#(v)} ∪ {LOW(w) | w a child of v} ∪ {dfs#(x) | {v,x} is a back edge from v} )

DFS(v) for Finding Articulation Points

Global initialization: v.dfs# = -1 for all v.
DFS(v)
  v.dfs# = dfscounter++
  v.low = v.dfs#
  // initialization
  for each edge {v,x}
    if (x.dfs# == -1) // x is undiscovered
      DFS(x)
      v.low = min(v.low, x.low)
    if (x.low >= v.dfs#)
      print "v is art. pt., separating x"
    else if (x is not v's parent)
      v.low = min(v.low, x.dfs#)

Equiv: "if {v,x} is a back edge" Why?