CSE 421
Algorithms
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Lecture 28
NP-Completeness

Populating the NP-Completeness Universe
• Circuit Sat \( \leq_p \) 3-SAT
• 3-SAT \( \leq_p \) Independent Set
• 3-SAT \( \leq_p \) Vertex Cover
• Independent Set \( \leq_p \) Clique
• 3-SAT \( \leq_p \) Hamiltonian Circuit
• Hamiltonian Circuit \( \leq_p \) Traveling Salesman
• 3-SAT \( \leq_p \) Integer Linear Programming
• 3-SAT \( \leq_p \) Graph Coloring
• 3-SAT \( \leq_p \) Subset Sum
• Subset Sum \( \leq_p \) Scheduling with Release times and deadlines

Cook’s Theorem
• The Circuit Satisfiability Problem is NP-Complete
• Circuit Satisfiability
  – Given a boolean circuit, determine if there is an assignment of boolean values to the input to make the output true

Proof of Cook’s Theorem
• Reduce an arbitrary problem Y in NP to X
• Let A be a non-deterministic polynomial time algorithm for Y
• Convert A to a circuit, so that Y is a Yes instance iff and only if the circuit is satisfiable

Satisfiability
• Given a boolean formula, does there exist a truth assignment to the variables to make the expression true

Circuit SAT
Find a satisfying assignment

Definitions

- **Boolean variable:** $x_1, \ldots, x_n$
- **Term:** $x_i$ or its negation $\neg x_i$
- **Clause:** disjunction of terms
  - $t_1$ or $t_2$ or $\ldots$ $t_j$
- **Problem:**
  - Given a collection of clauses $C_1, \ldots, C_k$, does there exist a truth assignment that makes all the clauses true
  - $(x_1$ or $\neg x_2)$, $(\neg x_1$ or $\neg x_3)$, $(x_2$ or $\neg x_3)$

3-SAT

- **Each clause has exactly 3 terms
- **Variables:** $x_1, \ldots, x_n$
- **Clauses $C_1, \ldots, C_k$**
  - $C_j = (t_{j1}$ or $t_{j2}$ or $t_{j3})$
- **Fact:** Every instance of SAT can be converted in polynomial time to an equivalent instance of 3-SAT

Find a satisfying truth assignment

$(x || y || z) \& \& (x || y || z) \& \& (x || y) \& \& (x || y) \& \& (y || z) \& \& (y || z)$

Theorem: CircuitSat $\leq_P$ 3-SAT

Theorem: 3-SAT $\leq_P$ IndSet

Sample Problems

- **Independent Set**
  - Graph $G = (V, E)$, a subset $S$ of the vertices is independent if there are no edges between vertices in $S$
**Vertex Cover**

- **Vertex Cover**
  - Graph $G = (V, E)$, a subset $S$ of the vertices is a vertex cover if every edge in $E$ has at least one endpoint in $S$.

**IS $\leq_P$ VC**

- Lemma: A set $S$ is independent iff $V-S$ is a vertex cover.
- To reduce IS to VC, we show that we can determine if a graph has an independent set of size $K$ by testing for a vertex cover of size $n - K$.

**Clique**

- **Clique**
  - Graph $G = (V, E)$, a subset $S$ of the vertices is a clique if there is an edge between every pair of vertices in $S$.

**Complement of a Graph**

- **Defn:** $G'=(V,E')$ is the complement of $G=(V,E)$ if $(u,v)$ is in $E'$ iff $(u,v)$ is not in $E$.
- Construct the complement.

**IS $\leq_P$ Clique**

- Lemma: $S$ is Independent in $G$ iff $S$ is a Clique in the complement of $G$.
- To reduce IS to Clique, we compute the complement of the graph. The complement has a clique of size $K$ iff the original graph has an independent set of size $K$. 

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**Find an maximum independent set $S$**

**Show that $V-S$ is a vertex cover**
Hamiltonian Circuit Problem

• Hamiltonian Circuit – a simple cycle including all the vertices of the graph

Thm: Hamiltonian Circuit is NP Complete

• Reduction from 3-SAT

Traveling Salesman Problem

• Given a complete graph with edge weights, determine the shortest tour that includes all of the vertices (visit each vertex exactly once, and get back to the starting point)

Thm: HC \leq_p TSP