Network Flow Definitions

- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, c(e) >= 0
- Problem, assign flows f(e) to the edges such that:
  - 0 <= f(e) <= c(e)
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
    - The flow leaving the source is as large as possible

Residual Graph

- Flow graph showing the remaining capacity
- Flow graph G, Residual Graph G_R
  - G: edge e from u to v with capacity c and flow f
  - G_R: edge e' from u to v with capacity c - f
  - G_R: edge e'' from v to u with capacity f

Augmenting Path Lemma

- Let P = v_1, v_2, ..., v_k be a path from s to t with minimum capacity b in the residual graph.
- b units of flow can be added along the path P in the flow graph.
Proof
• Add b units of flow along the path P
• What do we need to verify to show we have a valid flow after we do this?

Ford-Fulkerson Algorithm (1956)
while not done
    Construct residual graph G_R
    Find an s-t path P in G_R with capacity b > 0
    Add b units along in G

    If the sum of the capacities of edges leaving S is at most C, then the algorithm takes at most C iterations

Cuts in a graph
• Cut: Partition of V into disjoint sets S, T with s in S and t in T.
• Cap(S,T): sum of the capacities of edges from S to T
• Flow(S,T): net flow out of S
    – Sum of flows out of S minus sum of flows into S

• Flow(S,T) <= Cap(S,T)

What is Cap(S,T) and Flow(S,T)

Minimum value cut

Find a minimum value cut
MaxFlow – MinCut Theorem

• There exists a flow which has the same value of the minimum cut
• Proof: Consider a flow where the residual graph has no s-t path with positive capacity
• Let S be the set of vertices in $G_R$ reachable from s with paths of positive capacity

Max Flow - Min Cut Theorem

• Ford-Fulkerson algorithm finds a flow where the residual graph is disconnected, hence FF finds a maximum flow.

Performance

• The worst case performance of the Ford-Fulkerson algorithm is horrible

Better methods of finding augmenting paths

• Find the maximum capacity augmenting path
  – $O(m^2 \log(C))$ time
• Find the shortest augmenting path
  – $O(m^2n)$
• Find a blocking flow in the residual graph
  – $O(mn \log n)$