Minimum Cost Spanning Trees

Kruskal’s Algorithm:
Another Example of the Greedy Method

Min Cost Spanning Tree

- Given: Undirected graph $G = (V, E)$ & positive edge cost/weight $c(e) \in \mathbb{R}$ for each $e \in E$.
- Find: connected $T \subseteq E$ minimizing $\sum_{e \in T} c(e)$

Applications

- Broadcast tree in a network
- Building roads or power lines
- Routing power & ground on a PC board
- Clustering
- ...

Lemma 1: Trees and Cycles

Adding an edge to a tree creates a cycle; deleting any cycle edge gives a tree

- Corollary 1: Solution to MST is a tree
- Corollary 2: Cheapest edge in $E$ is in $T$

Exercise:

2nd-cheapest edge also in $T$

3rd-cheapest?

Kruskal’s Algorithm

sort edges

$K := \emptyset$

while $|K| < n-1$ do

$e :=$ next cheapest edge

if $K \cup \{e\}$ is acyclic

then $K := K \cup \{e\}$

else discard $e$

Correctness

Theorem: Kruskal’s algorithm builds an MST
Proof:

- Suppose Kruskal picks the tree $K$
- Suppose MST $M$ maximizes $|K \cap M|$ among all MSTs
- For sake of contradiction, suppose $K \neq M$
- Let $e$ be the cheapest edge in $K - M$
- Then…
Claim

M ∪ \{e\} has a cycle containing an edge f s.t.
(1) f ∉ K, and
(2) c(f) ≥ c(e)

Proof:
(1) If all the cycle edges were in K, then K wouldn’t be a tree.
(2) If c(f) < c(e), greedy looked at f before e. But \{e' ∈ K | c(e') < c(e)\} ∪ \{f\} ⊆ M, hence acyclic, so f would have been picked, but it wasn’t.

Correctness (cont.)

Theorem: Kruskal’s algorithm builds an MST

Proof:
- Suppose Kruskal picks the tree K
- Suppose MST M maximizes |K ∩ M| among all MSTs
- For sake of contradiction, suppose K∩M = ∅
- Let e be the cheapest edge in K – M
- From claim, ∃ f in cycle s.t. f ∉ K, c(f) ≥ c(e)
- Let M’ = (M ∪ \{e\}) – \{f\}
- Then M’ is an MST with |K ∩ M’| > |K ∩ M|.
- Contradiction. QED

Implementation

Testing “if K ∪ \{e\} is acyclic”:

Union/Find problem
- Linear space
- Time n α(n)
- α(n) < 5 for all n < age of universe