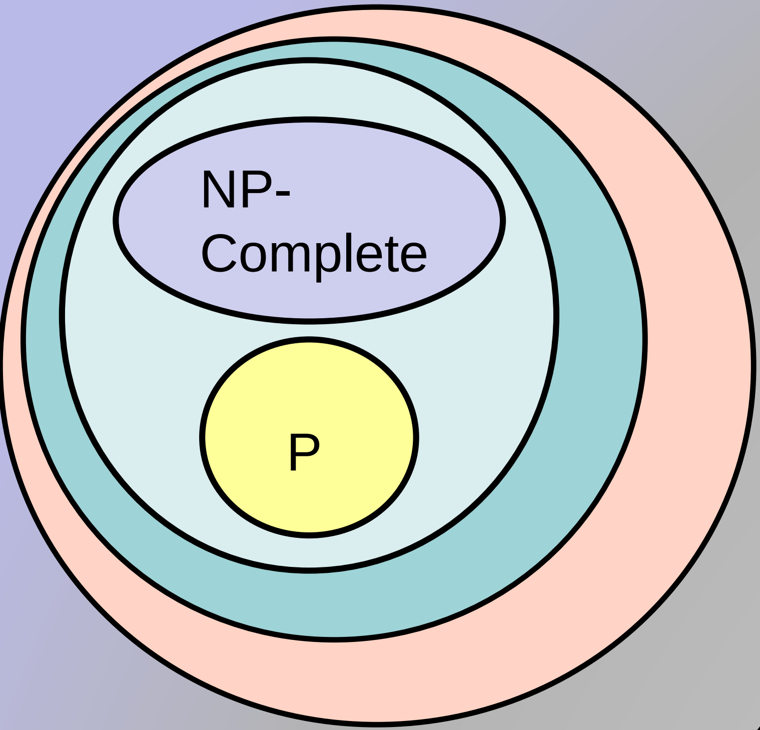
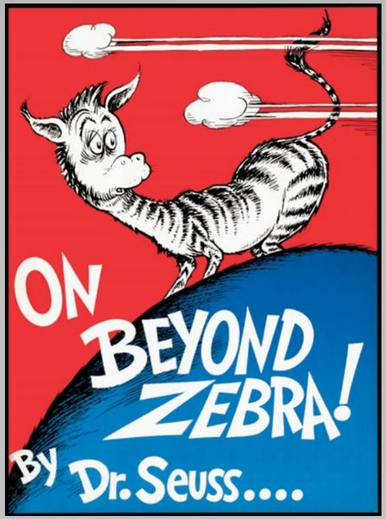




CSE 417
Algorithms and Complexity

Winter 2023

Lecture 26

NP-Completeness and Beyond

Announcements

- Final Exam: Monday, March 13, 8:30 AM
  - CSE2 G10, 1 hour 50 minutes, Closed Book
  - Comprehensive (but roughly 2/3rds post midterm)
  - Topics will include: recurrences, dynamic programming, graph algorithms, NP-Completeness

~~Fri, March 3~~	~~NP-Completeness: Overview, Definitions~~
~~Mon, March 6~~	~~NP-Completeness: Reductions~~
~~Wed, March 8~~	~~NP-Completeness: Problem Survey~~
Fri, March 10	Theory and Beyond NP-Completeness
Mon, March 13	Final Exam

NP-Completeness Proofs

- Prove that problem X is NP-Complete
  - Show that X is in NP (usually easy)
  - Pick a known NP complete problem Y
  - Show $Y <_P X$

Reducibility Among Combinatorial Problems

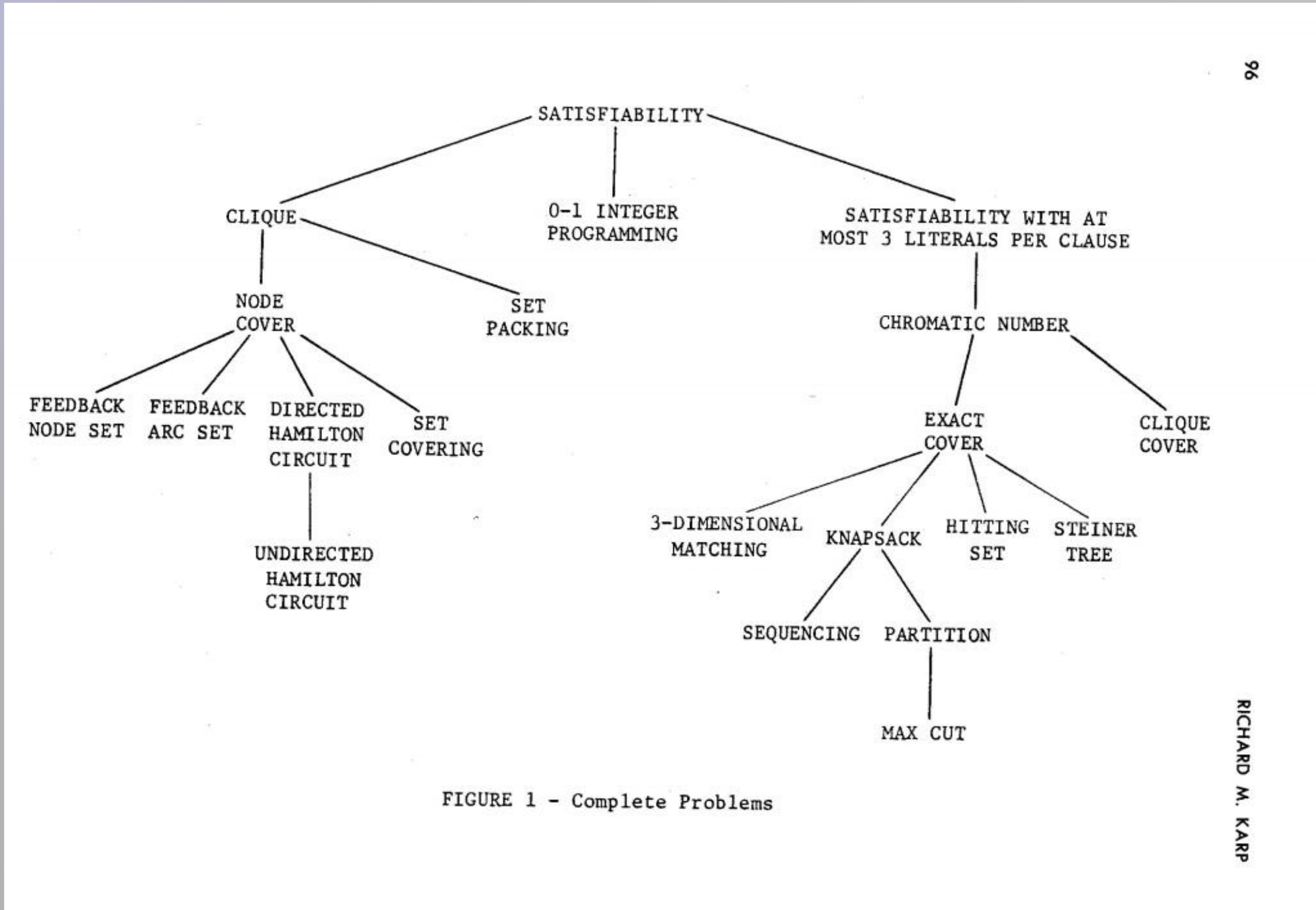
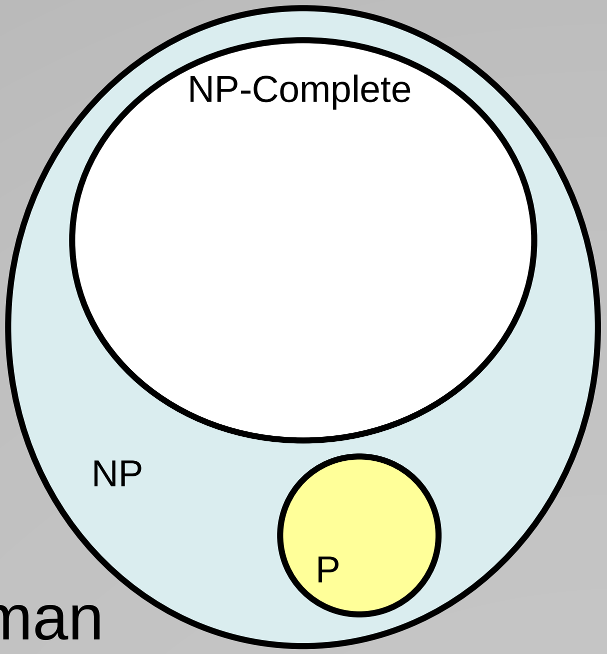


FIGURE 1 - Complete Problems

Populating the NP-Completeness Universe



- Circuit Sat $<_P$ 3-SAT
- 3-SAT $<_P$ Independent Set
- 3-SAT $<_P$ Vertex Cover
- Independent Set $<_P$ Clique
- 3-SAT $<_P$ Hamiltonian Circuit
- Hamiltonian Circuit $<_P$ Traveling Salesman
- 3-SAT $<_P$ Integer Linear Programming
- 3-SAT $<_P$ Graph Coloring
- 3-SAT $<_P$ Subset Sum
- Subset Sum $<_P$ Scheduling with Release times and deadlines

Coping with NP-Completeness

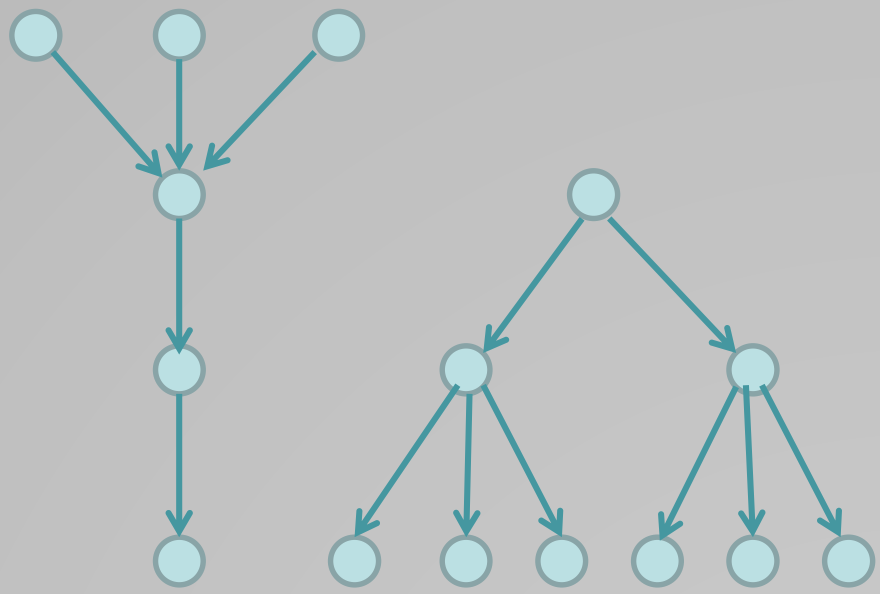
- Approximation Algorithms
- Exact solution via Branch and Bound
- Local Search

I can't find an efficient algorithm, but neither can all these famous people.

Multiprocessor Scheduling

- Unit execution tasks
- Precedence graph
- K-Processors

- Polynomial time for k=2
- Open for k = constant
- NP-complete is k is part of the problem



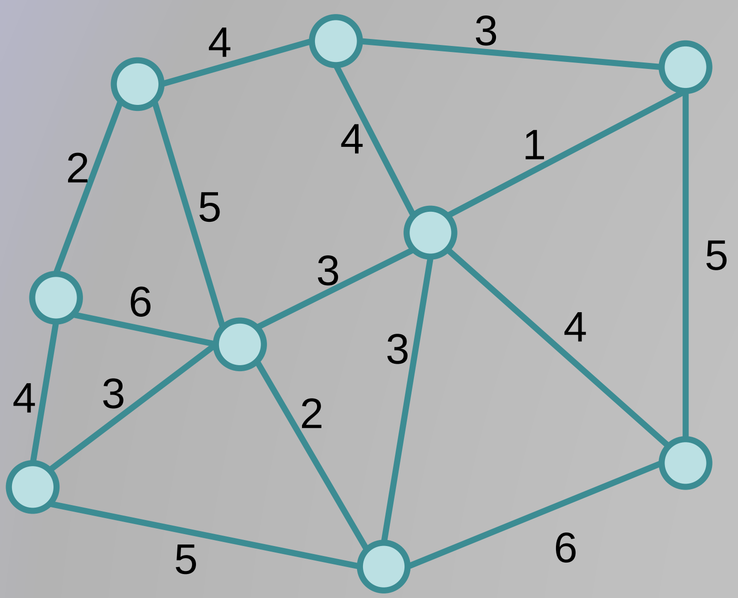
Highest level first is 2-Optimal

Choose k items on the highest level

Claim: number of rounds is at least twice the optimal.

Christofides TSP Algorithm

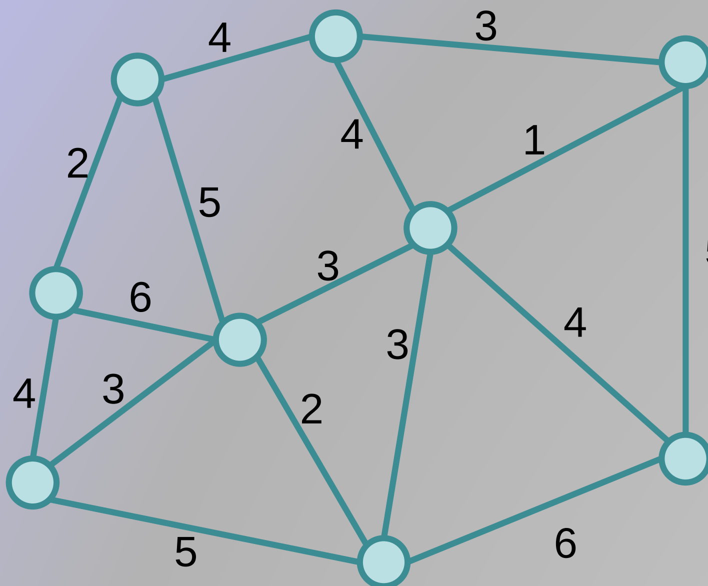
- Undirected graph satisfying triangle inequality

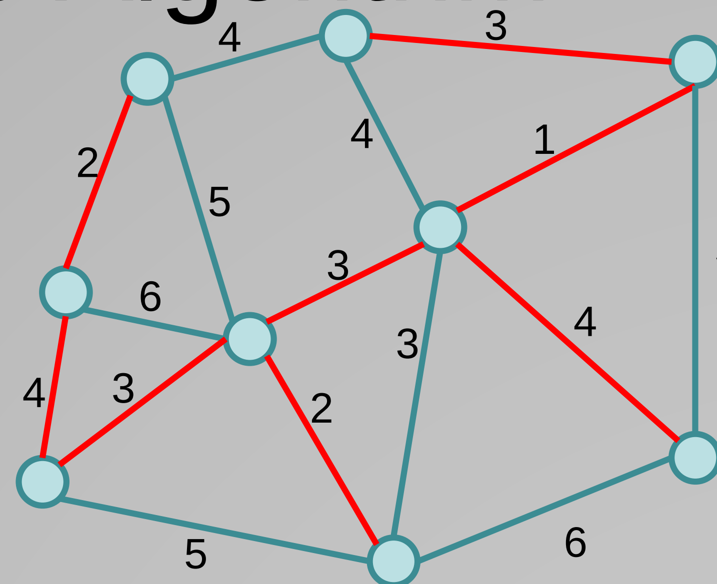


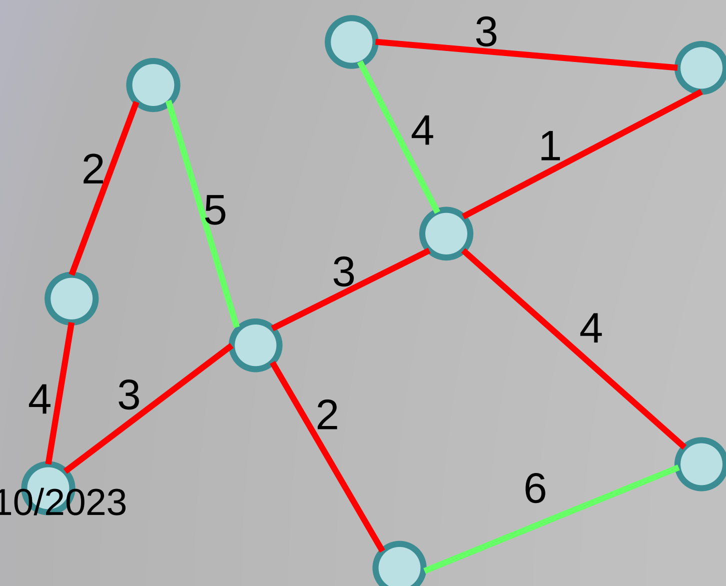
1. Find MST
2. Add additional edges so that all vertices have even degree
3. Build Eulerian Tour

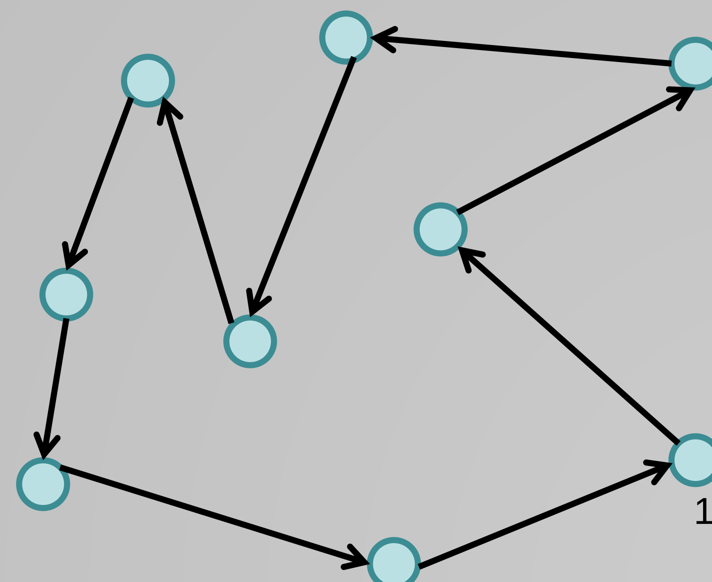
3/2 Approximation

Christofides Algorithm







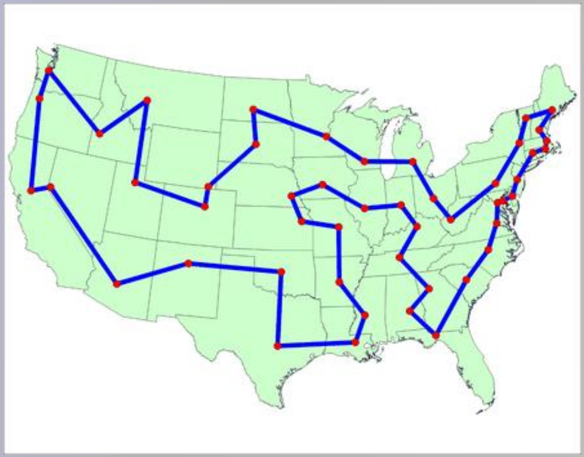


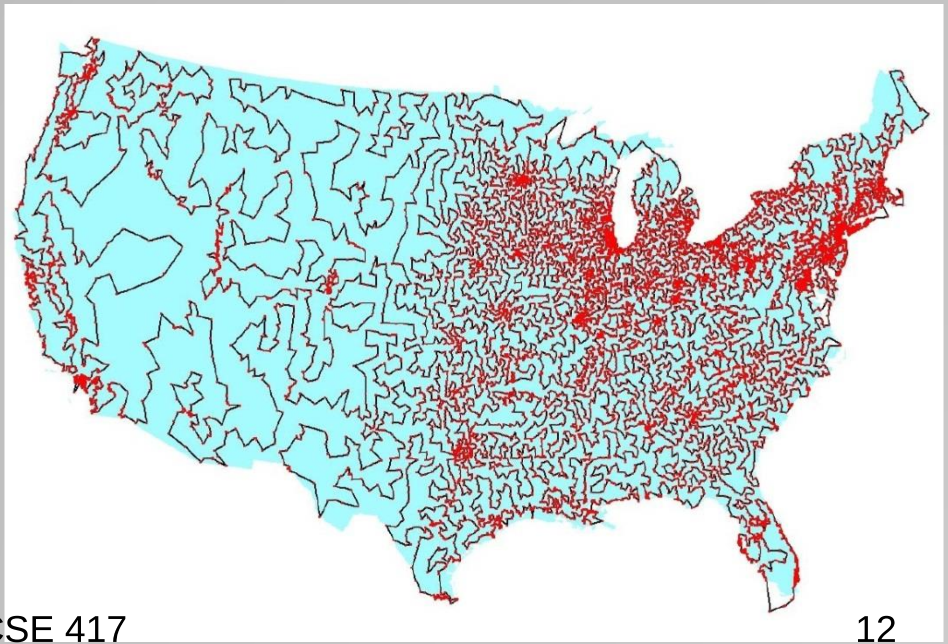
Branch and Bound

- Brute force search – tree of all possible solutions
- Branch and bound – compute a lower bound on all possible extensions
  - Prune sub-trees that cannot be better than optimal

Branch and Bound for TSP

- Enumerate all possible paths
- Lower bound, Current path cost plus MST of remaining points
- Euclidean TSP
  - Points on the plane with Euclidean Distance
  - Sample data set: State Capitals



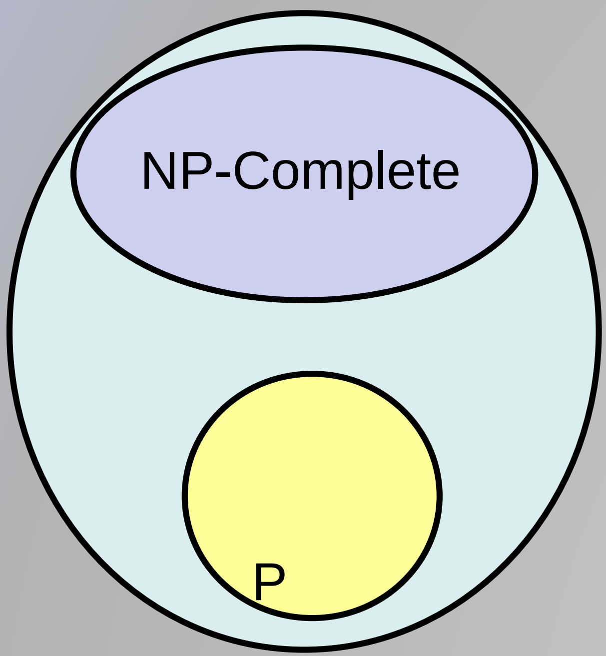


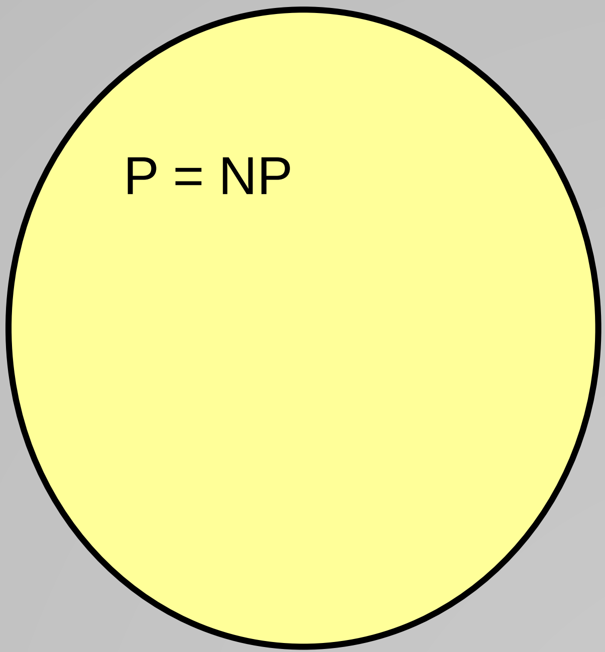
Local Optimization

- Improve an optimization problem by local improvement
  - Neighborhood structure on solutions
  - Travelling Salesman 2-Opt (or K-Opt)
  - Independent Set Local Replacement

What we don't know

- P vs. NP

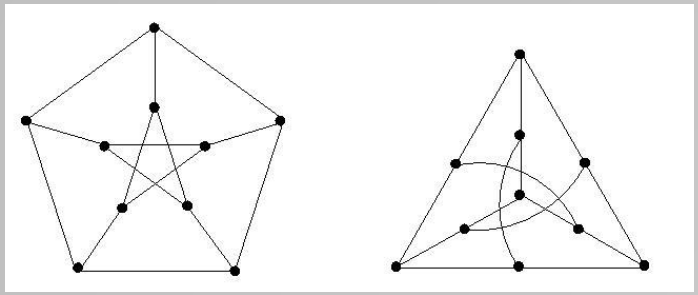




If P $\neq$ NP, is there anything in between

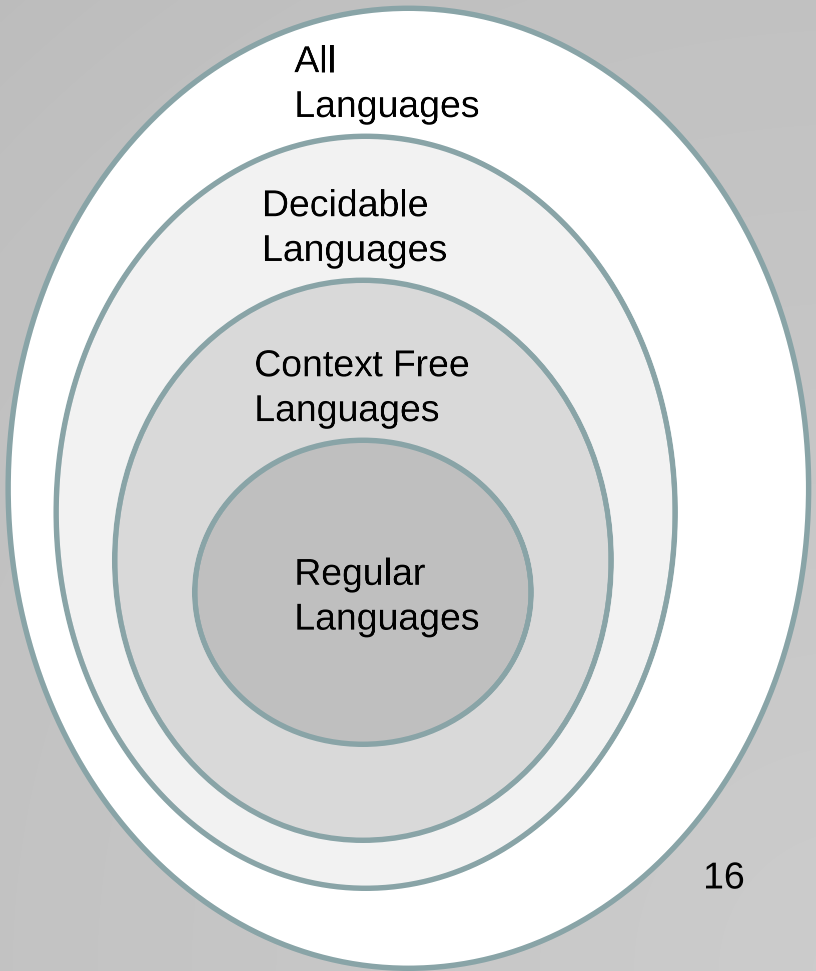
- Yes, Ladner [1975]
- Problems not known to be in P or NP Complete
  - Factorization
  - Discrete Log
  - Graph Isomorphism

Solve $g^k = b$ over a finite group



Complexity Theory

- Computational requirements to recognize languages
- Models of Computation
- Resources
- Hierarchies



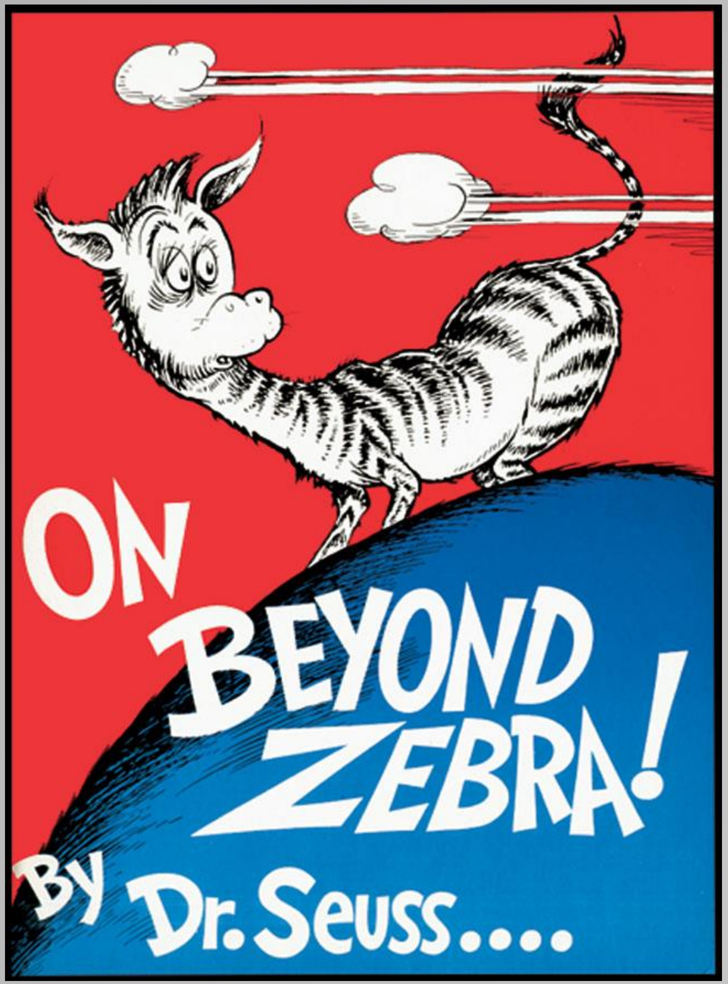
Time complexity

- P: (Deterministic) Polynomial Time
- NP: Non-deterministic Polynomial Time
- EXP: Exponential Time

Space Complexity

- Amount of Space (Exclusive of Input)
- L: Logspace, problems that can be solved in O(log n) space for input of size n
  - Related to Parallel Complexity
- PSPACE, problems that can be required in a polynomial amount of space

So what is beyond NP?



NP vs. Co-NP

- Given a Boolean formula, is it true for some choice of inputs

- Given a Boolean formula, is it true for all choices of inputs

Problems beyond NP

- Exact TSP, Given a graph with edge lengths and an integer K, does the minimum tour have length K

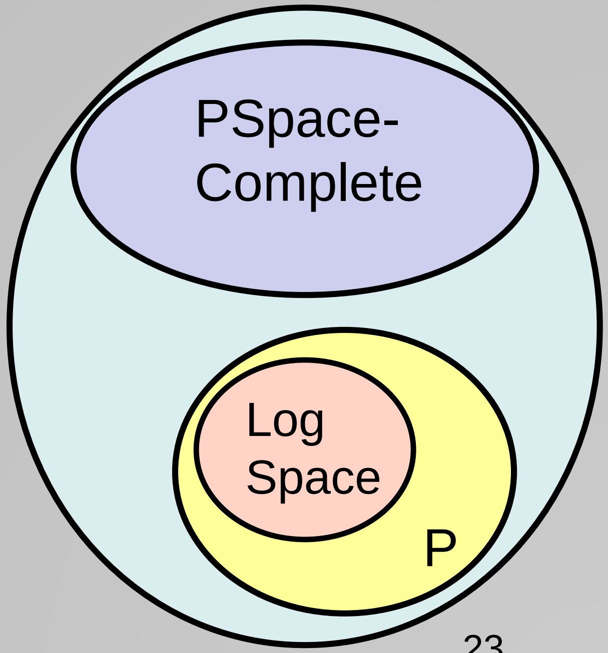
- Minimum circuit, Given a circuit C, is it true that there is no smaller circuit that computes the same function a C

Polynomial Hierarchy

- Level 1
  - $\exists X_1\ \Phi(X_1)$, $\forall X_1\ \Phi(X_1)$
- Level 2
  - $\forall X_1 \exists X_2\ \Phi(X_1,X_2)$, $\exists X_1 \forall X_2\ \Phi(X_1,X_2)$
- Level 3
  - $\forall X_1 \exists X_2 \forall X_3\ \Phi(X_1,X_2,X_3)$, $\exists X_1 \forall X_2 \exists X_3\ \Phi(X_1,X_2,X_3)$

Polynomial Space

- Quantified Boolean Expressions
  - $\exists X_1 \forall X_2 \exists X_3 ... \exists X_{n-1} \forall X_n \; \Phi(X_1, X_2, X_3 \ldots X_{n-1} X_n)$
- Space bounded games
  - Competitive Facility Location Problem
  - N x N Chess
- Counting problems
  - The number of Hamiltonian Circuits



N X N Chess

Even Harder Problems

```
public int[] RecolorSwap(int[] coloring) {
        int k = maxColor(coloring);

        for (int v = 0; v < nVertices; v++) {
            if (coloring[v] == k) {
                int[] nbdColorCount = ColorCount(v, k, coloring);
                List<Edge> edges1 = vertices[v].Edges;

                foreach (Edge e1 in edges1) {
                    int w = e1.Head;
                    if (nbdColorCount[coloring[w]] == 1)
                        if (RecolorSwap(v, w, k, coloring))
                            break;
                }
            }
        }
        return coloring;
    }
```

Is this code correct?

Halting Problem

- Given a program P that does not take any inputs, does P eventually exit?

Impossibility of solving the Halting Problem

Suppose Halt(P) returns true if P halts, and false otherwise

Consider the program G:

```
Define G {
    if (Halt(G)){
        while (true) ;
    }
    else {
        exit();
    }
}
```

What is Halt(G)?

Undecidable Problems

- The Halting Problem is undecidable
- Impossible problems are hard . . .