

Lecture25

CSE 417

Algorithms and Complexity

Winter 2023

Lecture 25

NP-Completeness, Part III

3/8/2023

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Announcements

- Homework 9
- Exam practice problems on course homepage
- Final Exam: Monday, March 13, 8:30 AM

Fri, March 3	NP-Completeness: Overview, Definitions
Mon, March 6	NP-Completeness: Reductions
Wed, March 8	NP-Completeness: Problem Survey
Fri, March 10	Theory and Beyond NP-Completeness
Mon, March 13	Final Exam

NP Completeness: The story so far

Circuit Satisfiability is
NP-Complete



There are a whole bunch of
other important problems
which are NP-Complete



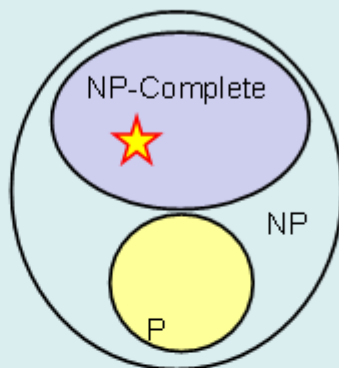
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Cook's Theorem

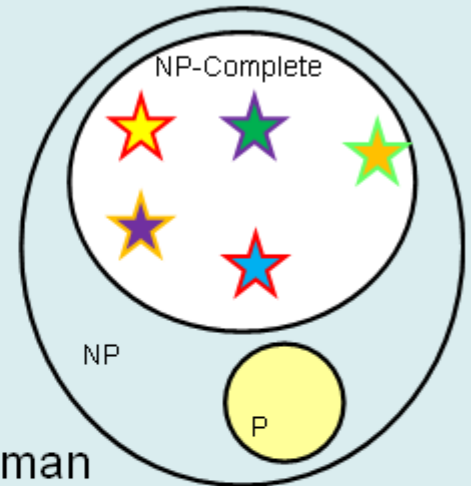
- Definition:
 - X is NP-Complete if:
 - X is in NP
 - For all Z in NP: $Z \leq_p X$
- There is an NP Complete problem
 - The Circuit Satisfiability Problem



∴
 if $X \in \text{NP-Complete}$
 $\Rightarrow \text{NP-Complete}$
 and $X \in P$
 then $P = \text{NP}$

Populating the NP-Completeness Universe

- Circuit Sat \leq_P 3-SAT
- 3-SAT \leq_P Independent Set
- 3-SAT \leq_P Vertex Cover
- Independent Set \leq_P Clique
- 3-SAT \leq_P Hamiltonian Circuit
- Hamiltonian Circuit \leq_P Traveling Salesman
- 3-SAT \leq_P Integer Linear Programming
- 3-SAT \leq_P Graph Coloring
- 3-SAT \leq_P 3 Dimensional Matching
- 3-SAT \leq_P Subset Sum
- Subset Sum \leq_P Scheduling with Release times and deadlines



Satisfiability

Literal: A Boolean variable or its negation.

$$x_i \text{ or } \bar{x}_i$$

Clause: A disjunction of literals.

$$C_j = x_1 \vee \bar{x}_2 \vee x_3$$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses.

$$\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$$

SAT: Given CNF formula Φ , does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

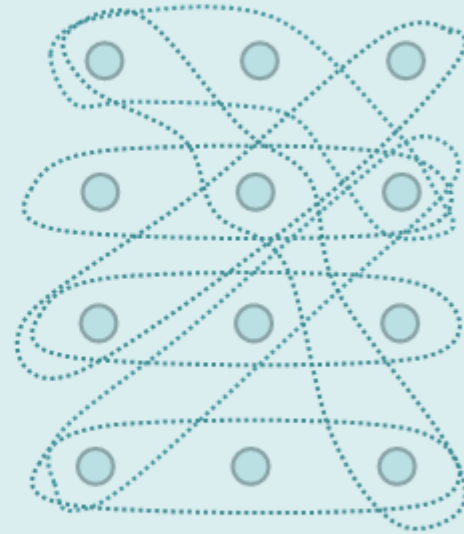
Ex: $(\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

Yes: $x_1 = \text{true}, x_2 = \text{true}, x_3 = \text{false}$.

Matching

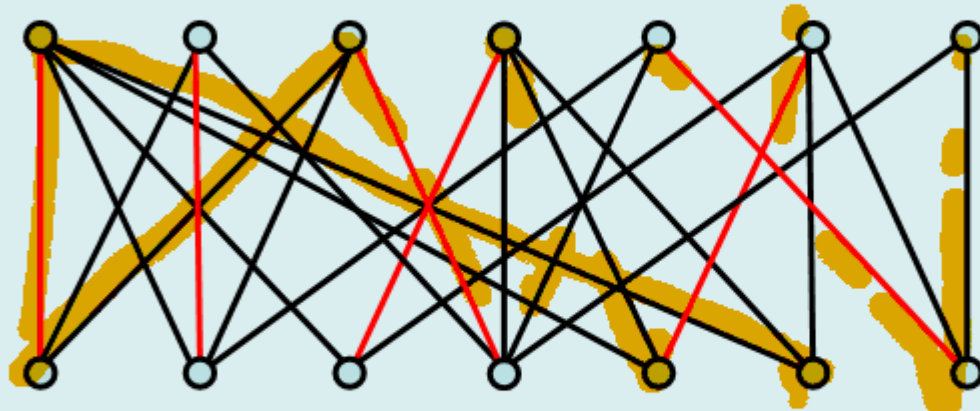


Two dimensional matching



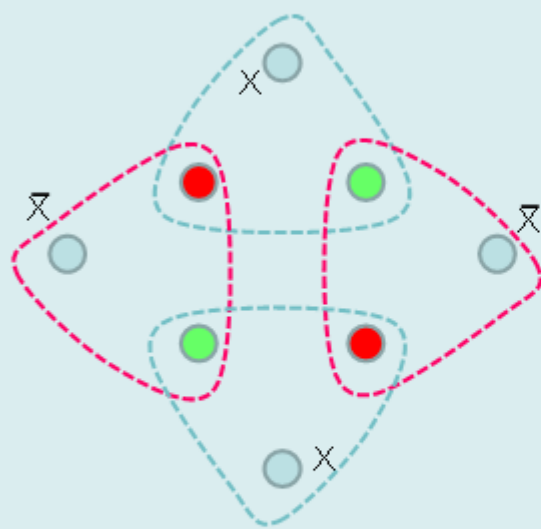
Three dimensional matching (3DM)

Augmenting Path Algorithm for Matching

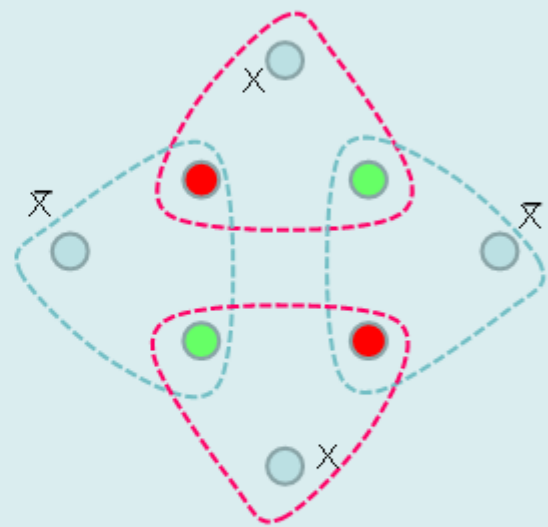


Find augmenting path in $O(m)$ time
n phases of augmentation
 $O(nm)$ time algorithm for matching

3-SAT \leq_P 3DM



X True



X False

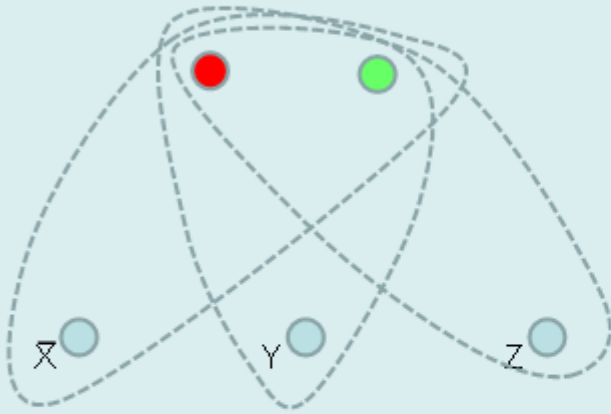
Truth Setting Gadget

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3-SAT \leq_P 3DM



Clause gadget for $(\bar{X} \text{ OR } Y \text{ OR } Z)$



Garbage Collection Gadget
(Many copies)

X.C.2

2DM



X U

Exact Cover (sets of size 3) XC3

Given a collection of sets of size 3 of a domain of size $3N$, is there a sub-collection of N sets that cover the sets

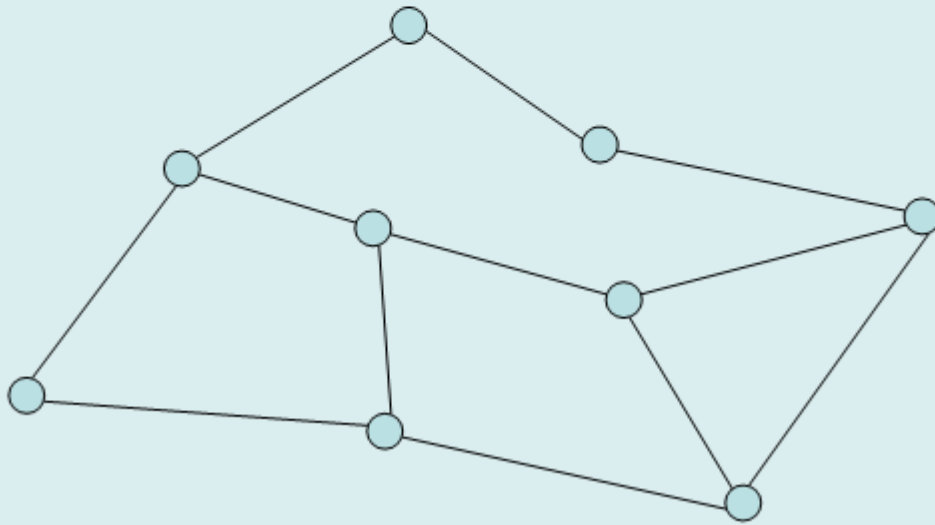
(A, B, C), (D, E, F), (A, B, G),
 (A, C, I), (B, E, G), (A, G, I),
 (B, D, F), (C, E, I), (C, D, H),
 (D, G, I), (D, F, H), (E, H, I),
 (F, G, H), (F, H, I)



$$3DM \leq_P XC3$$

Graph Coloring

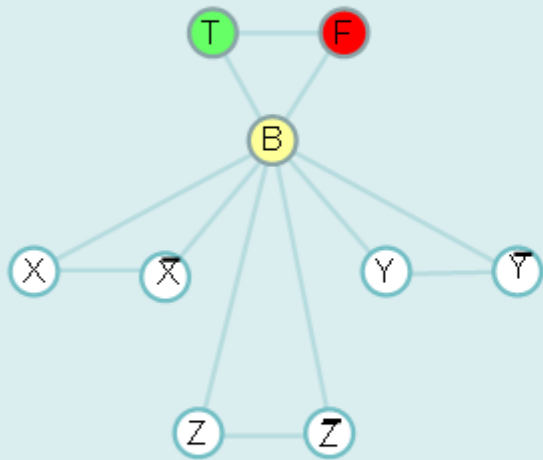
- NP-Complete
 - Graph K-coloring
 - Graph 3-coloring
- Polynomial
 - Graph 2-Coloring



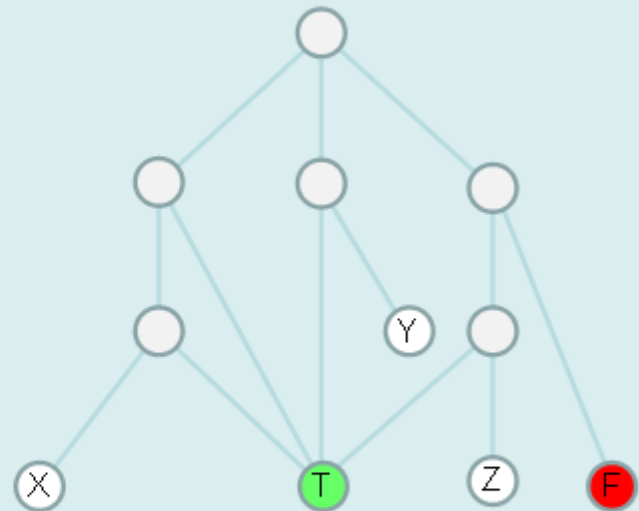
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3-SAT \leq_p 3 Colorability



Truth Setting Gadget



Clause Testing Gadget

(Can be colored if at least one input is T)

Number Problems

- Subset sum problem
 - Given natural numbers w_1, \dots, w_n and a target number W , is there a subset that adds up to exactly W ?
- Subset sum problem is NP-Complete
- Subset Sum problem can be solved in $O(nW)$ time

$$W = 123456789 \quad (234)$$

$$50781234$$

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XC3 $<_p$ SUBSET SUM

Idea: Represent each set as a large integer, where the element x_i is encoded as D^i where D is an integer

$$\{x_3, x_5, x_9\} \Rightarrow D^3 + D^5 + D^9$$

Does there exist a subset that sums to exactly $D^1 + D^2 + D^3 + \dots + D^{n-1} + D^n$

Detail: How large is D ? We need to make sure that we do not have any carries, so we can choose $D = m+1$, where m is the number of sets.

105010100

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11111111

Integer Linear Programming

- Linear Programming – maximize a linear function subject to linear constraints
- Integer Linear Programming – require an integer solution
- NP Completeness reduction from 3-SAT

Use 0-1 variables for x_i 's

Constraint for clause $(x_1 \vee \overline{x_2} \vee \overline{x_3})$

$$x_1 + (1 - x_2) + (1 - x_3) > 0$$

Linear Programming
 max $2x + 5y$



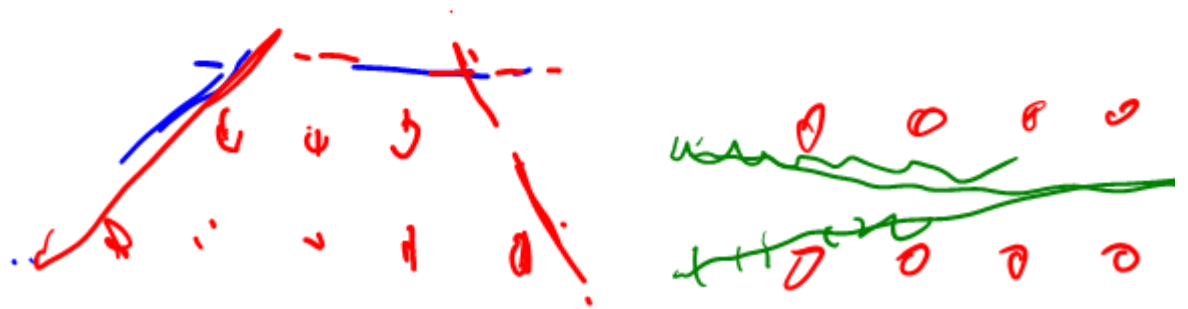
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 y

$$x + y \leq 3$$

$$x + 4y \leq 11$$

$$3x + 2y \leq 9$$

Integer Linear Programming.



Scheduling with release times and deadlines (RD-Sched)

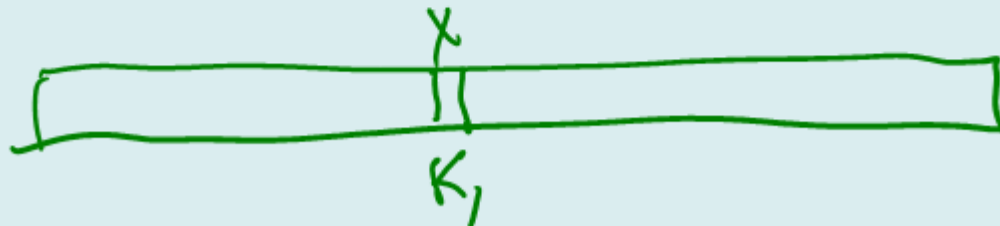
- Tasks, $\{t_1, t_2, \dots, t_n\}$
- Task t_j has a length l_j , release time r_j and deadline d_j
- Once a task is started, it is worked on without interruption until it is completed
- Can all tasks be completed satisfying constraints?

Subset Sum $<_p$ RD-Sched

- Subset Sum Problem
 - $\{s_1, s_2, \dots, s_N\}$, integer K_1
 - Does there exist a subset that sums to K_1 ?
 - Assume the total sums to K_2

Reduction

- Tasks $\{t_1, t_2, \dots, t_N, x\}$
- t_j has length s_j , release 0, deadline $K_2 + 1$
- x has length 1, release K_1 , deadline $K_1 + 1$



Friday: NP-Completeness and Beyond!

