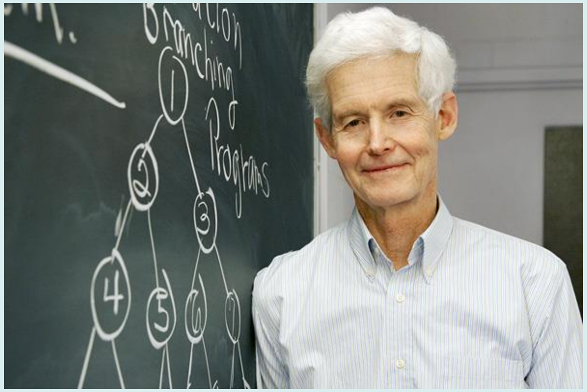
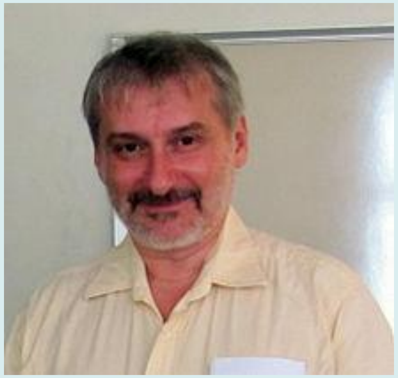


CSE 417
Algorithms and Complexity

Winter 2023

Lecture 25

NP-Completeness, Part III

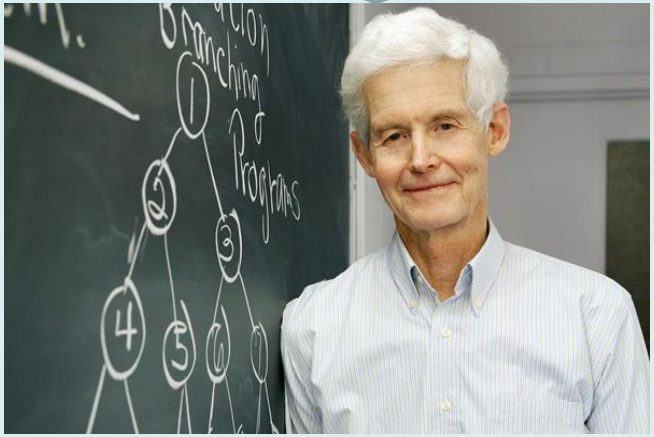
Announcements

- Homework 9
- Exam practice problems on course homepage
- Final Exam: Monday, March 13, 8:30 AM

~~Fri, March 3~~	~~NP-Completeness: Overview, Definitions~~
~~Mon, March 6~~	~~NP-Completeness: Reductions~~
Wed, March 8	NP-Completeness: Problem Survey
Fri, March 10	Theory and Beyond NP-Completeness
Mon, March 13	Final Exam

NP Completeness:
The story so far

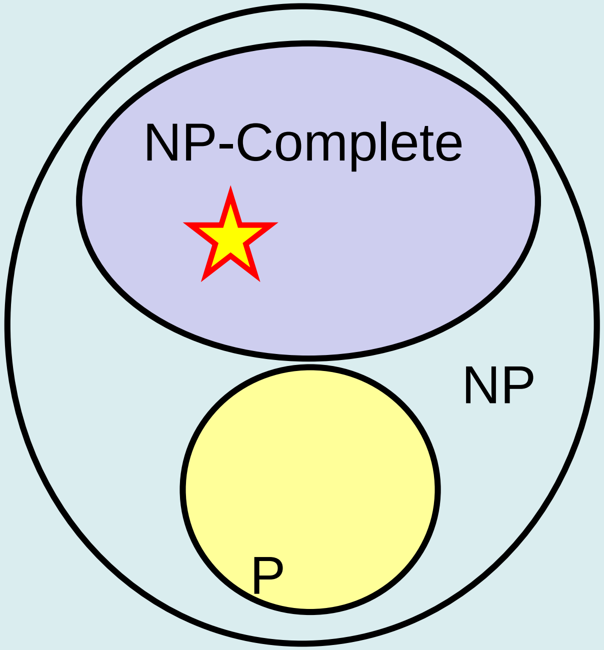
Circuit Satisfiability is NP-Complete



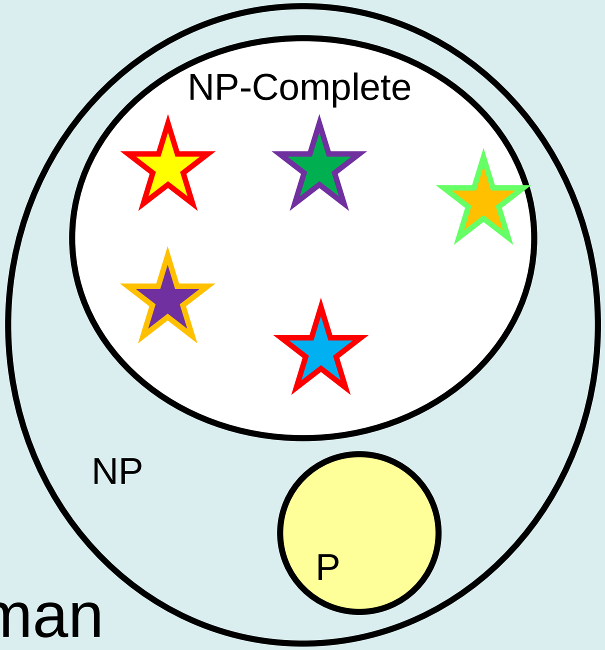
There are a whole bunch of other important problems which are NP-Complete

Cook's Theorem

- Definition:
  - X is NP-Complete if:
    - X is in NP
    - For all Z in NP: $Z <_P X$
- There is an NP Complete problem
  - The Circuit Satisfiability Problem



Populating the NP-Completeness Universe



- Circuit Sat $<_P$ 3-SAT
- 3-SAT $<_P$ Independent Set
- 3-SAT $<_P$ Vertex Cover
- Independent Set $<_P$ Clique
- 3-SAT $<_P$ Hamiltonian Circuit
- Hamiltonian Circuit $<_P$ Traveling Salesman
- 3-SAT $<_P$ Integer Linear Programming
- 3-SAT $<_P$ Graph Coloring
- 3-SAT $<_P$ 3 Dimensional Matching
- 3-SAT $<_P$ Subset Sum
- Subset Sum $<_P$ Scheduling with Release times and deadlines

Satisfiability

Literal: A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause: A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

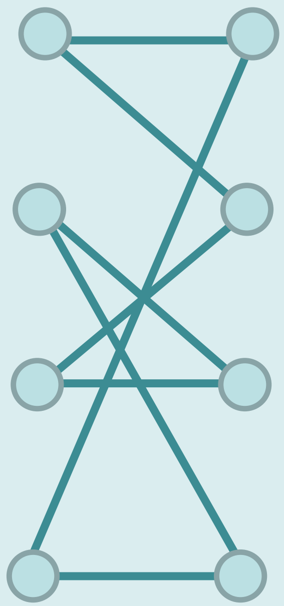
SAT: Given CNF formula Φ, does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

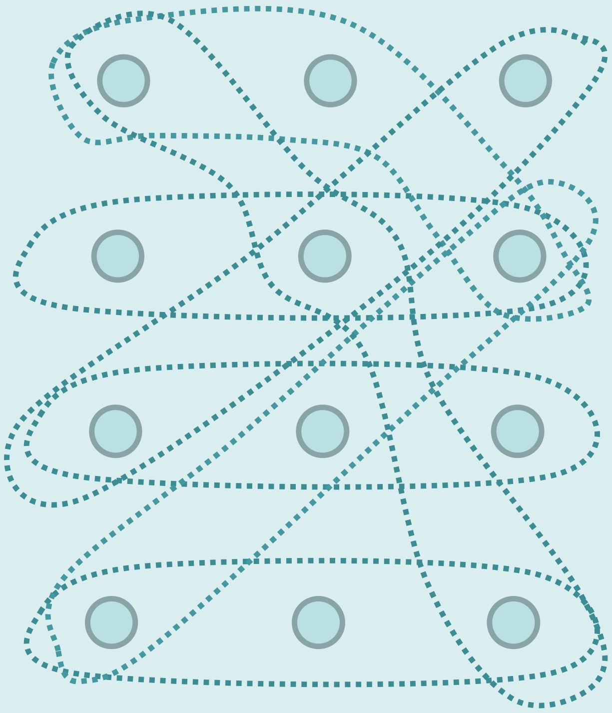
Ex: $\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_2 \vee x_3\right) \wedge \left(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}\right)$

Yes: x_1 = true, x_2 = true x_3 = false.

Matching

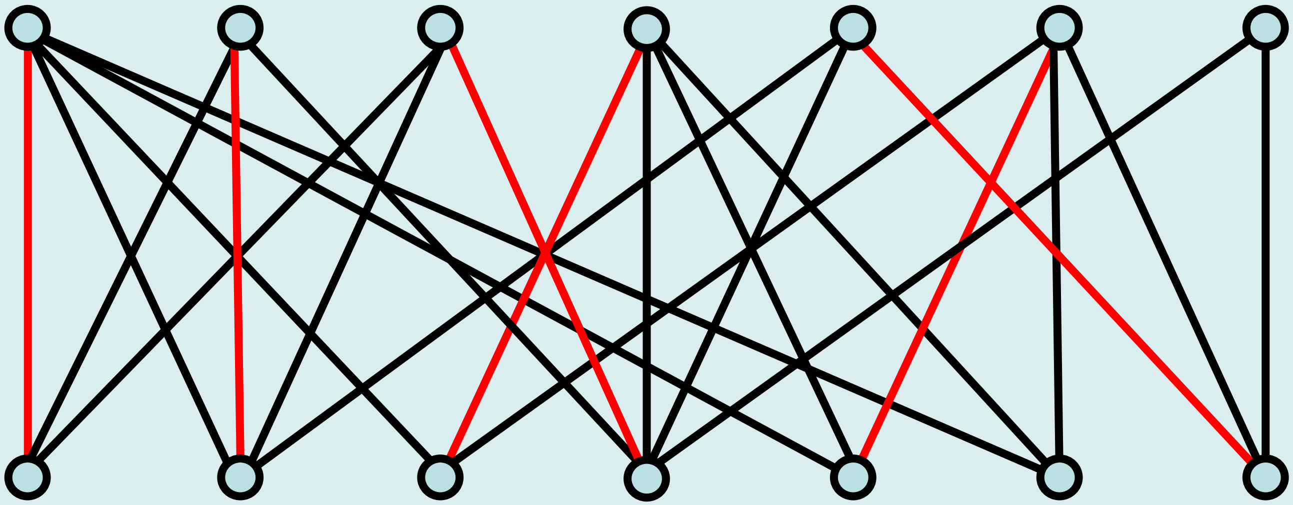


Two dimensional matching



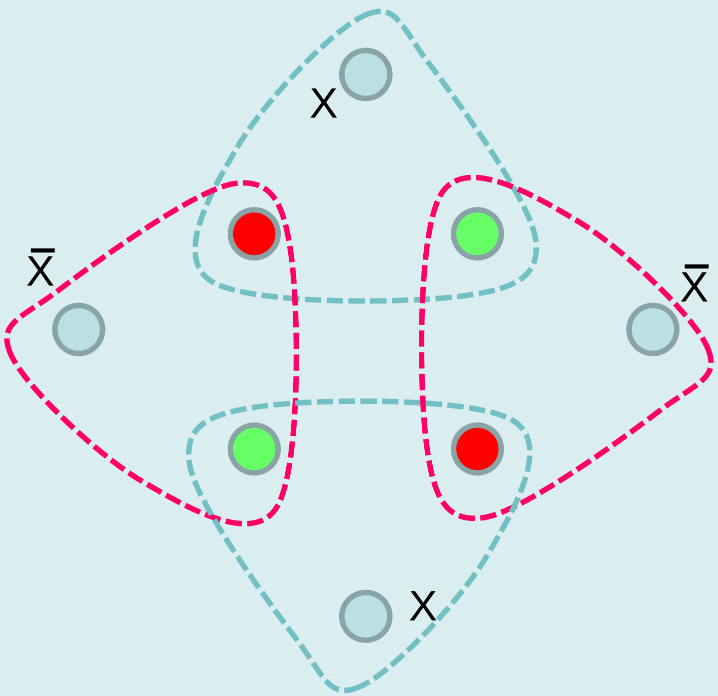
Three dimensional matching (3DM)

Augmenting Path Algorithm for Matching

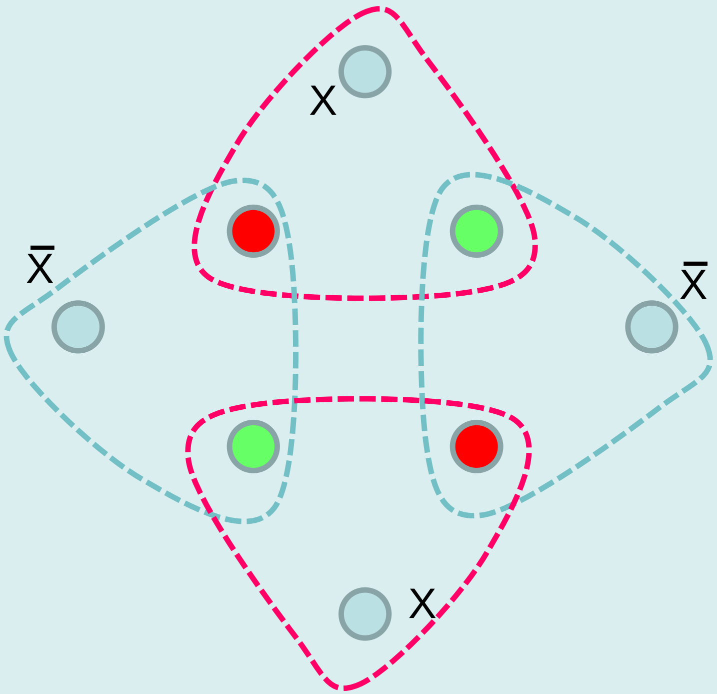


Find augmenting path in O(m) time
n phases of augmentation
O(nm) time algorithm for matching

3-SAT $<_P$ 3DM



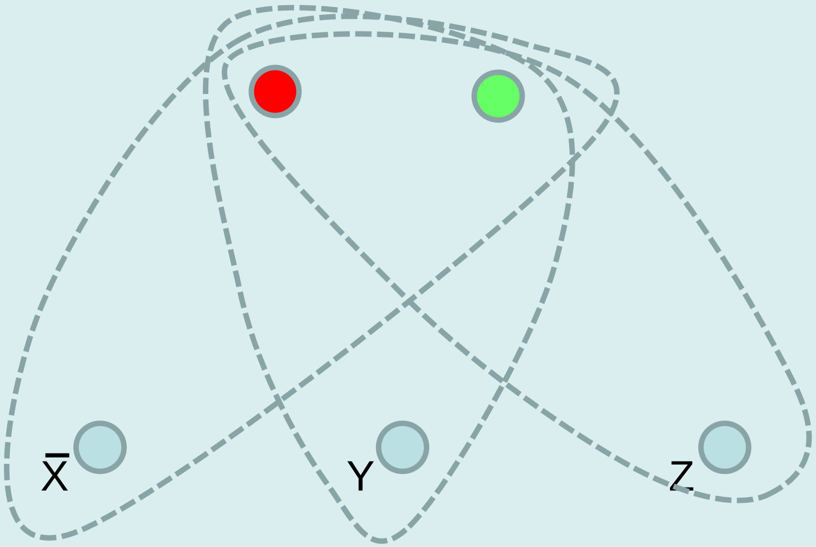
X True



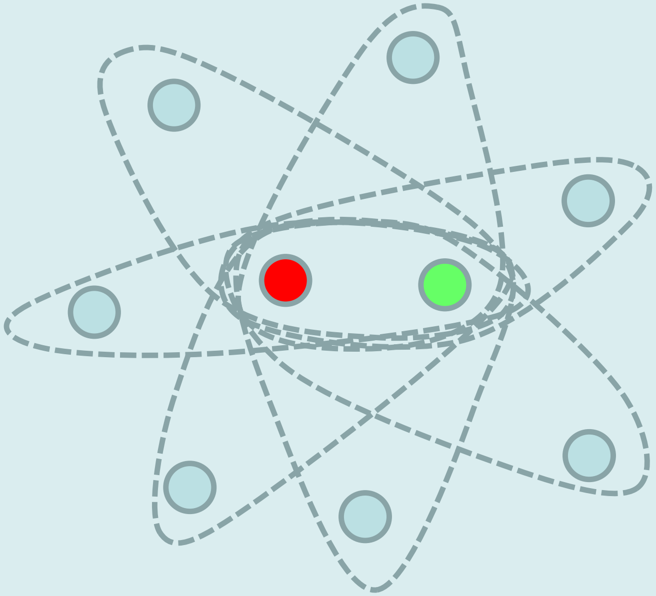
X False

Truth Setting Gadget

3-SAT $<_P$ 3DM



Clause gadget for (X̄ OR Y OR Z)



Garbage Collection Gadget
(Many copies)

Exact Cover (sets of size 3) XC3

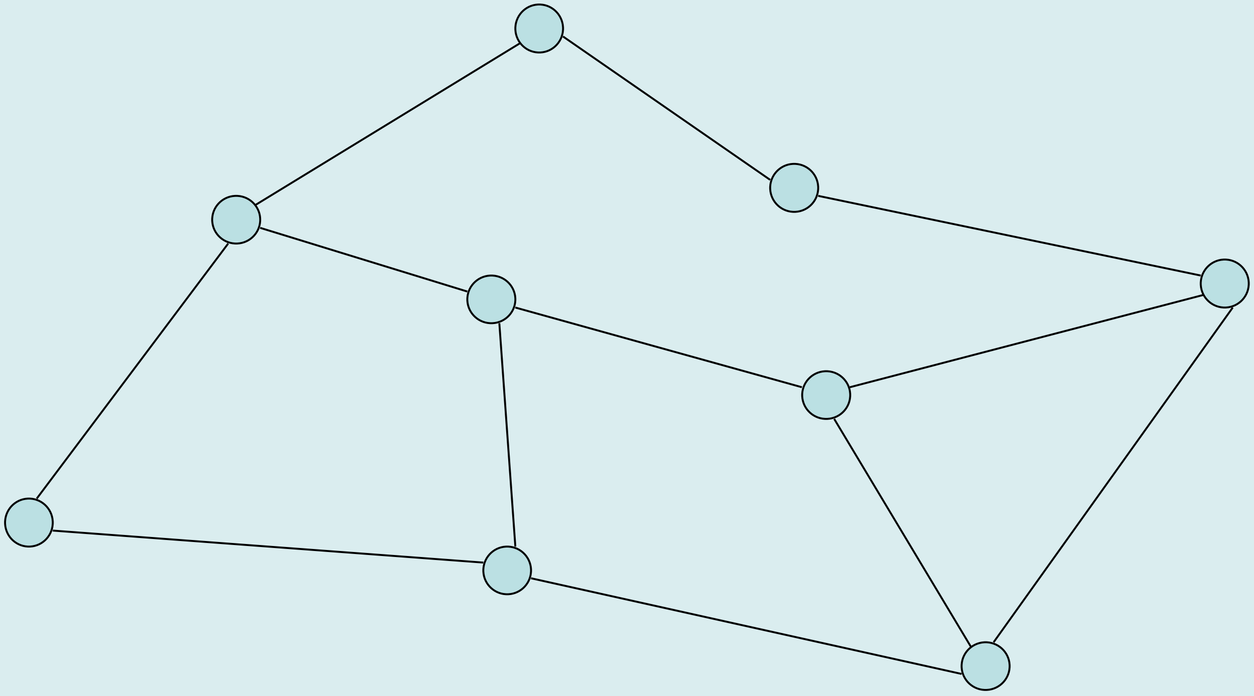
Given a collection of sets of size 3 of a domain of size 3N, is there a sub-collection of N sets that cover the sets

(A, B, C), (D, E, F), (A, B, G),
(A, C, I), (B, E, G), (A, G, I),
(B, D, F), (C, E, I), (C, D, H),
(D, G, I), (D, F, H), (E, H, I),
(F, G, H), (F, H, I)

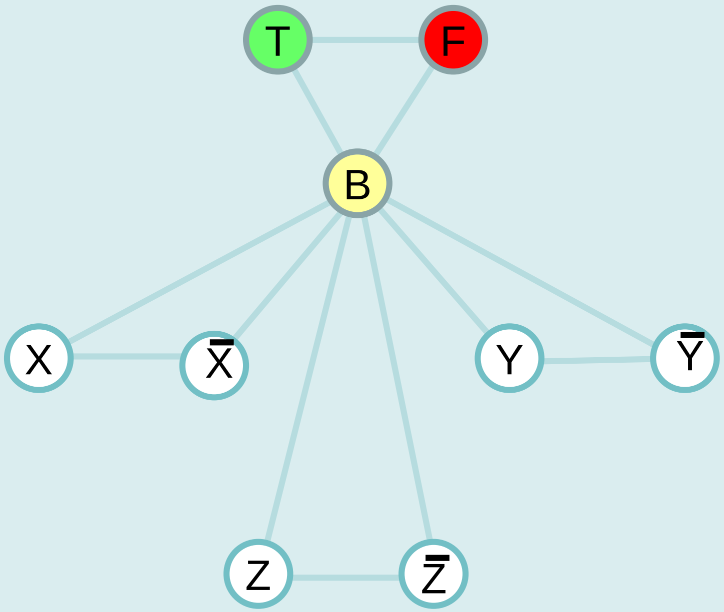
$$3DM <_P XC3$$

Graph Coloring

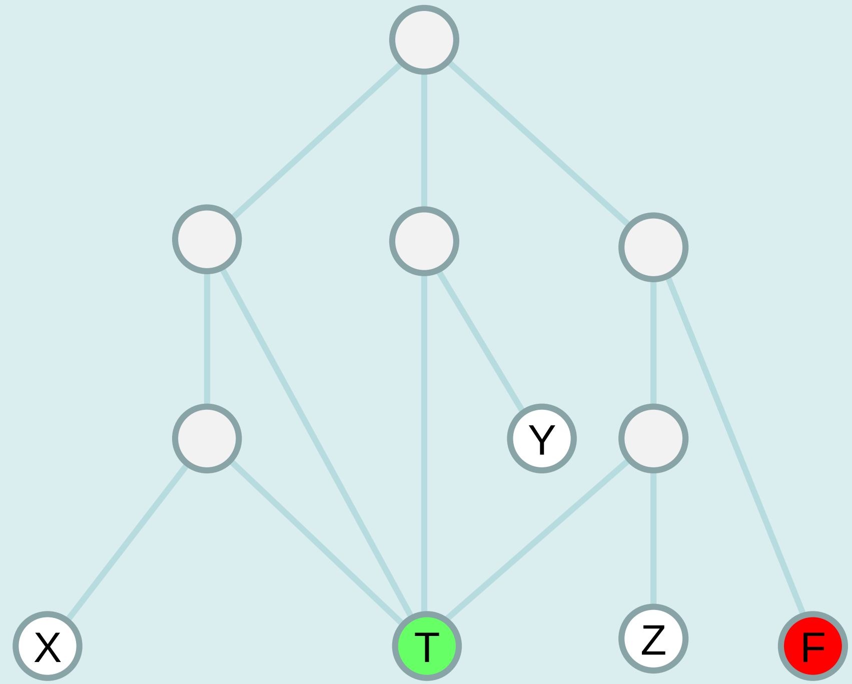
- NP-Complete
  - Graph K-coloring
  - Graph 3-coloring
- Polynomial
  - Graph 2-Coloring



3-SAT $<_P$ 3 Colorability



Truth Setting Gadget



Clause Testing Gadget

(Can be colored if at least one input is T)

Number Problems

- Subset sum problem
  - Given natural numbers $w_1,. . ., w_n$ and a target number W, is there a subset that adds up to exactly W?

- Subset sum problem is NP-Complete
- Subset Sum problem can be solved in O(nW) time

XC3 $<_P$ SUBSET SUM

Idea: Represent each set as a large integer, where the element x_i is encoded as D^i where D is an integer

$\{x_3, x_5, x_9\} \Rightarrow D^3 + D^5 + D^9$

Does there exist a subset that sums to exactly
$D^1 + D^2 + D^3 + \ldots + D^{n-1} + D^n$

Detail: How large is D? We need to make sure that we do not have any carries, so we can choose D = m+1, where m is the number of sets.

Integer Linear Programming

- Linear Programming – maximize a linear function subject to linear constraints
- Integer Linear Programming – require an integer solution
- NP Completeness reduction from 3-SAT

Use 0-1 variables for x_i's

Constraint for clause: $(x_1 \vee \overline{x_2} \vee \overline{x_2})$

$x_1 + (1 - x_2) + (1 - x_3) > 0$

Scheduling with release times and deadlines (RD-Sched)

- Tasks, $\{t_1, t_2, \ldots t_n\}$
- Task t_j has a length l_j, release time r_j and deadline d_j
- Once a task is started, it is worked on without interruption until it is completed
- Can all tasks be completed satisfying constraints?

Subset Sum $<_P$ RD-Sched

- Subset Sum Problem
  - $\{s_1, s_2, . . . , s_N\}$, integer K_1
  - Does there exist a subset that sums to K_1?
  - Assume the total sums to K_2

Reduction

- Tasks $\{t_1, t_2, \ldots t_N, x\}$
- t_j has length s_j, release 0, deadline $K_2 + 1$
- x has length 1, release K_1, deadline $K_1 + 1$

Friday: NP-Completeness and Beyond!