CSE 417
Algorithms and Complexity

Winter 2023

Lecture 24

NP-Completeness, Part II

Announcements

- Homework 9
- Exam practice problems on course homepage
- Final Exam: Monday, March 13, 8:30 AM

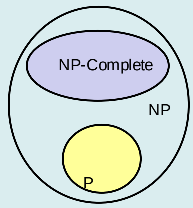
~~Fri, March 3~~	~~NP-Completeness: Overview, Definitions~~
Mon, March 6	NP-Completeness: Reductions
Wed, March 8	NP-Completeness: Problem Survey
Fri, March 10	Theory and Beyond NP-Completeness
Mon, March 13	Final Exam

Key Idea: Problem Reduction

- Use an algorithm for problem X to solve problem Y.
  - This means that problem X is more difficult that problem Y
- Terminology: Y is reducible to X
  - Notation: Y $<_P$ X

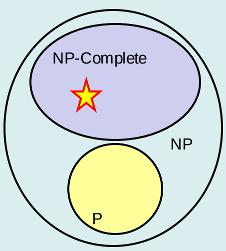
The Universe

- P: Polynomial Time
- NP: Nondeterministic Polynomial Time
  - Problems where a "yes" answer can be verified in polynomial time
- NP-Complete
  - The hardest problems in NP

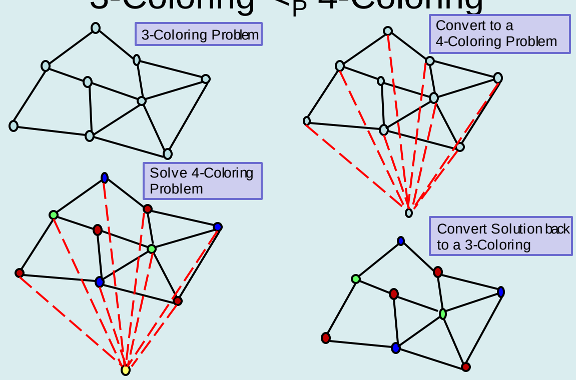


NP-Completeness

- If X is NP-Complete, Z is in NP and X $<_P$ Z
  - Z is NP-Complete
- Steve Cook got this started by finding the first NP-Complete problem

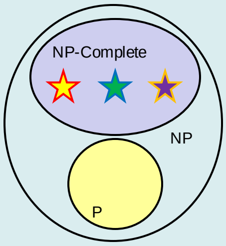


3-Coloring $<_P$ 4-Coloring



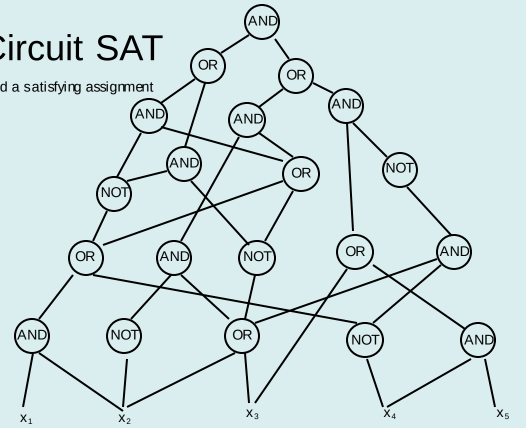
Starting NP-Complete Problems

- Circuit Satisfiability Problem
  - Cook's Theorem
- Boolean Formula Satisfiability
  - 3-SAT
- Maximum Independent Set



Circuit SAT

Find a satisfying assignment



Satisfiability

Literal: A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause: A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT: Given CNF formula Φ, does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

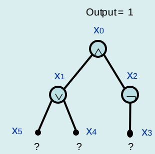
Ex: $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

Yes: x_1 = true, x_2 = true x_3 = false.

3-SAT is NP-Complete

Circuit SAT $<_P$ 3-SAT

Convert a circuit into a formula

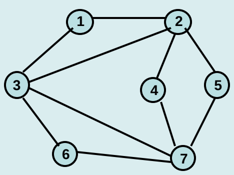


Each gate is represented by a set of clauses

$$(x_1 \vee x_1 \vee x_1) \wedge (x_2 \vee x_2 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_1 \vee x_1 \vee \overline{x_4}) \wedge (x_1 \vee x_1 \vee \overline{x_5}) \wedge (\overline{x_1} \vee x_4 \vee x_5) \wedge (\overline{x_0} \vee \overline{x_0} \vee x_1) \wedge (\overline{x_0} \vee \overline{x_0} \vee x_2) \wedge (x_0 \vee \overline{x_1} \vee \overline{x_2})$$

Independent Set

- Independent Set
  - Graph G = (V, E), a subset S of the vertices is independent if there are no edges between vertices in S



3 Satisfiability Reduces to Independent Set

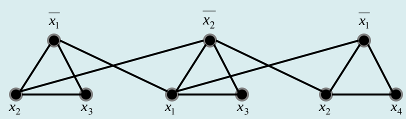
Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G

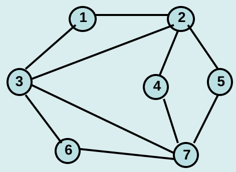


k = 3

$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$

Vertex Cover

- Vertex Cover
  - Graph G = (V, E), a subset S of the vertices is a vertex cover if every edge in E has at least one endpoint in S

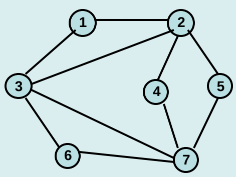


IS $<_P$ VC

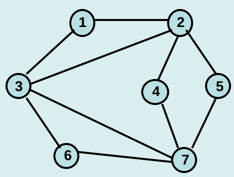
- Lemma: A set S is independent iff V-S is a vertex cover
- To reduce IS to VC, we show that we can determine if a graph has an independent set of size K by testing for a Vertex cover of size n - K

IS $<_P$ VC

Find a maximum independent set S

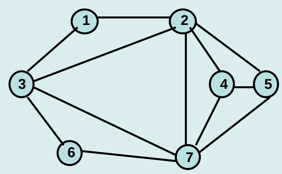


Show that V-S is a vertex cover



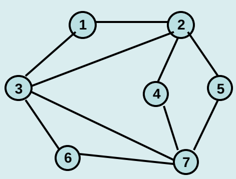
Clique

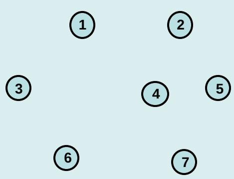
- Clique
  - Graph G = (V, E), a subset S of the vertices is a clique if there is an edge between every pair of vertices in S



Complement of a Graph

- Defn: G'=(V,E') is the complement of G=(V,E) if (u,v) is in E' iff (u,v) is not in E



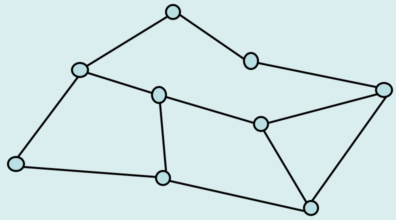


IS $<_P$ Clique

- Lemma: S is Independent in G iff S is a Clique in the complement of G
- To reduce IS to Clique, we compute the complement of the graph. The complement has a clique of size K iff the original graph has an independent set of size K

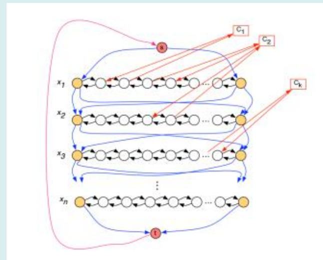
Hamiltonian Circuit Problem

- Hamiltonian Circuit – a simple cycle including all the vertices of the graph



Thm: Hamiltonian Circuit is NP Complete

- Reduction from 3-SAT



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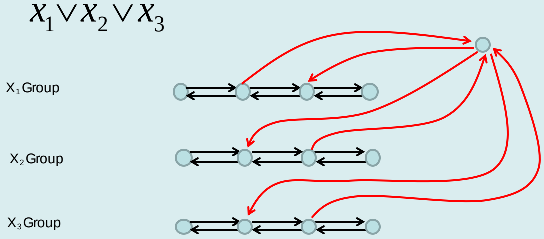
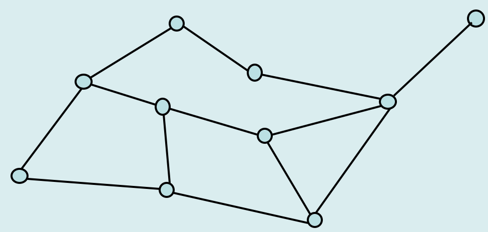
Clause Gadget

$$x_1 \lor \overline{x_2} \lor \overline{x_3}$$



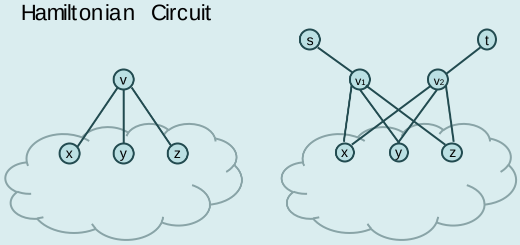
Hamiltonian Path Problem

- Hamiltonian Path – a simple path including all the vertices of the graph



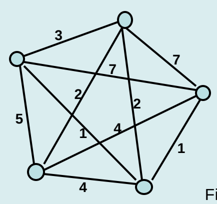
Reduce Hamiltonian Circuit to Hamiltonian Path

G_2 has a Hamiltonian Path iff G_1 has a Hamiltonian Circuit



Traveling Salesman Problem

- Given a complete graph with edge weights, determine the shortest tour that includes all of the vertices (visit each vertex exactly once, and get back to the starting point)



Find the minimum cost tour

Thm: HC $<_P$ TSP