# CSE 417
# Algorithms and Complexity

Winter 2023
Lecture 17
Divide and Conquer



## Announcements



## Divide and Conquer

- Algorithm paradigm
  - Break problems into subproblems until easy to solve
  - Work is split between "divide", "combine", and "base'' components
- Standard examples
  - MergeSort and QuickSort
- Analysis tool: Recurrences



## Matrix Multiplication

- N X N Matrix, A B = C

```
for (int i = 0; i < n; i++)
        for (int j = 0;  j < n; j++) {
                int t = 0;
                for (int k = 0; k < n; k++)
                        t = t + A[i][k] * B[k][j];
                C[i][j] = t;
        }
```



## Recursive Matrix Multiplication

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

r = ae + bf
s = ag + bh
t = ce + df
u = cg + dh

A N x N matrix can be viewed as a 2 x 2 matrix with entries that are (N/2) x (N/2) matrices.

The recursive matrix multiplication algorithm recursively multiplies the (N/2) x (N/2) matrices and combines them using the equations for multiplying 2 x 2 matrices



## Recursive Matrix Multiplication

- How many recursive calls are made at each level?
- How much work in combining the results?
- What is the recurrence?

## What is the run time for the recursive Matrix Multiplication Algorithm?

- Recurrence:



## Strassen's Algorithm

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = p_1 + p_2 - p_4 + p_6$

$s = p_4 + p_5$

$t = p_6 + p_7$

$u = p_2 - p_3 + p_5 - p_7$

Where:

$p_1 = (b - d)(f + h)$

$p_2 = (a + d)(e + h)$

$p_3 = (a - c)(e + g)$

$p_4 = (a + b)h$

$p_5 = a(g - h)$

$p_6 = d(f - e)$

$p_7 = (c + d)e$

From Aho, Hopcroft, Ullman 1974



## Recurrence for Strassen's Algorithms

- $T(n) = 7\,T(n/2) + cn^2$
- What is the runtime?

$\log_2 7 = 2.8073549221$



## Strassen's Algorithms

- Treat n x n matrices as 2 x 2 matrices of n/2 x n/2 submatrices
- Use Strassen's trick to multiply 2 x 2 matrices with 7 multiplies
- Base case standard multiplication for single entries
- Recurrence: $T(n) = 7\,T(n/2) + cn^2$
- Solution is $O(7^{\log n}) = O(n^{\log 7})$ which is about $O(n^{2.807})$



## Inversion Problem

- Let $a_1, \ldots a_n$ be a permutation of 1 . . n
- $(a_i, a_j)$ is an inversion if $i < j$ and $a_i > a_j$

4, 6, 1, 7, 3, 2, 5

- Problem: given a permutation, count the number of inversions
- This can be done easily in $O(n^2)$ time
  - Can we do better?



## Application

- Counting inversions can be use to measure how close ranked preferences are
  - People rank 20 movies, based on their rankings you cluster people who like that same type of movie

# Counting Inversions

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 12 | 4 | 1 | 7 | 2 | 3 | 15 | 9 | 5 | 16 | 8 | 6 | 13 | 10 | 14 |

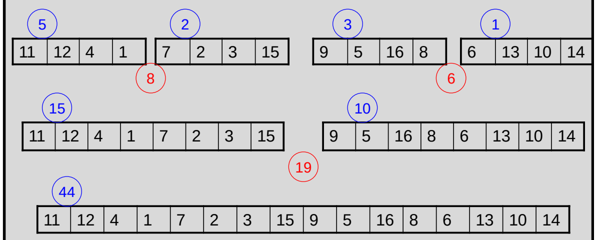
Count inversions on lower half

Count inversions on upper half

Count the inversions between the halves



# Count the Inversions






Problem – how do we count inversions between sub problems in O(n) time?

- Solution – Count inversions while merging

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 7 | 11 | 12 | 15 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5 | 6 | 8 | 9 | 10 | 13 | 14 | 16 |

Standard merge algorithm – add to inversion count when an element is moved from the upper array to the solution



# Use the merge algorithm to count inversions

| | | | |
|---|---|---|---|
| 1 | 4 | 11 | 12 |

| | | | |
|---|---|---|---|
| 2 | 3 | 7 | 15 |

| | | | |
|---|---|---|---|
| 5 | 8 | 9 | 16 |

| | | | |
|---|---|---|---|
| 6 | 10 | 13 | 14 |

Indicate the number of inversions for each element detected when merging



# Inversions

- Counting inversions between two sorted lists
  - O(1) per element to count inversions

x x x x x x x x

y y y y y y y y

z z z z z z z z z z z z z z z z

- Algorithm summary
  - Satisfies the "Standard recurrence"
  - T(n) = 2 T(n/2) + cn



# Computing the Median

- Given n numbers, find the number of rank n/2
- One approach is sorting
  - Sort the elements, and choose the middle one
  - Can you do better?
- *Selection*, given n numbers and an integer k, find the k-th largest

## Select(A, k)

```
Select(A, k){
        Choose element x from A
        S1 = {y in A | y < x}
        S2 = {y in A | y > x}
        S3 = {y in A | y = x}
        if (|S2| >= k)
                return Select(S2, k)
        else if (|S2| + |S3| >= k)
                return x
        else
                return Select(S1, k - |S2| - |S3|)
}
```

| $S_1$ | $S_3$ | $S_2$ |
|---|---|---|

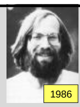


## Deterministic Selection

- What is the run time of select if we can guarantee that choose finds an x such that $|S_1| < 3n/4$ and $|S_2| < 3n/4$ in O(n) time
- What is the run time of select if we can guarantee that choose finds an x such that $|S_1| < 3n/4$ and $|S_2| < 3n/4$ in O(n) time



## BFPRT Algorithm





- A very clever choose algorithm . . .
- Deterministic algorithm that guarantees that $|S_1| < 3n/4$ and $|S_2| < 3n/4$
- Actual recurrence is:

$$T(n) \le T(3n/4) + T(n/5) + c\,n$$



## Integer Arithmetic

```
    9715480283945084383094856701043643845790217965702956767
+   1242431098234099057329075097179898430928779579277597977
```

Runtime for standard algorithm to add two n digit numbers:

```
    2095067093034680994318596846868779409766717133476767930
X   5920175091777634709677679342929097012308956679993010921
```

Runtime for standard algorithm to multiply two n digit numbers:



## Recursive Multiplication Algorithm (First attempt)

$x = x_1 2^{n/2} + x_0$

$y = y_1 2^{n/2} + y_0$

$xy = (x_1 2^{n/2} + x_0)(y_1 2^{n/2} + y_0)$

$\quad = x_1y_1 2^n + (x_1y_0 + x_0y_1)2^{n/2} + x_0y_0$

Recurrence:

Run time:



## Simple algebra

$x = x_1 2^{n/2} + x_0$

$y = y_1 2^{n/2} + y_0$

$xy = x_1y_1 2^n + (x_1y_0 + x_0y_1) 2^{n/2} + x_0y_0$

$p = (x_1 + x_0)(y_1 + y_0) = x_1y_1 + x_1y_0 + x_0y_1 + x_0y_0$

# Karatsuba's Algorithm

Multiply n-digit integers x and y

Let $x = x_1 2^{n/2} + x_0$ and $y = y_1 2^{n/2} + y_0$
Recursively compute
$a = x_1 y_1$
$b = x_0 y_0$
$p = (x_1 + x_0)(y_1 + y_0)$
Return $a2^n + (p - a - b)2^{n/2} + b$

Recurrence: $T(n) = 3T(n/2) + cn$

$\log_2 3 = 1.58496250073\ldots$