CSE 417
Algorithms and Complexity

Winter 2023
Lecture 16
Divide and Conquer and Recurrences

Divide and Conquer

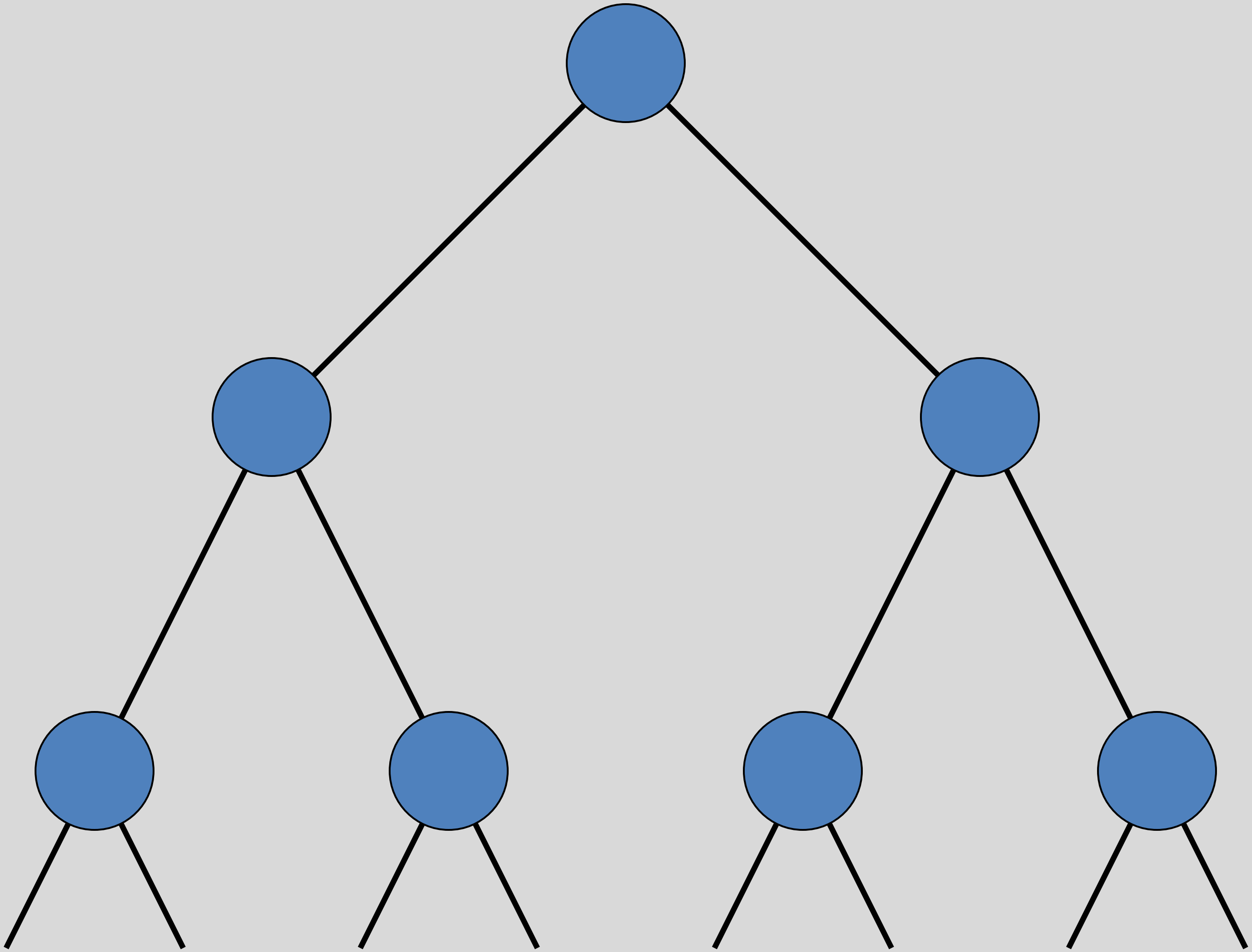
- Recurrences, Sections 5.1 and 5.2
- Algorithms
  - Median (Selection)
  - Fast Matrix Multiplication
  - Counting Inversions (5.3)
  - Multiplication (5.5)

Divide and Conquer : Merge Sort

```
Array MSort(Array a, int n){
    if (n <= 1)  return a;
    return Merge(MSort(a[0 .. n/2], n/2),  MSort(a[n/2+1 .. n-1], n/2);
}
```

$T(n) = 2T(n/2) + n; T(1) = 1;$

Unrolling the recurrence



A better mergesort (?)

- Divide into 3 subarrays and recursively sort
- Apply 3-way merge

What is the recurrence?

Unroll recurrence for $T(n) = 3T(n/3) + n$

$$T(n) = aT(n/b) + f(n)$$

$$T(n) = T(n/2) + cn$$

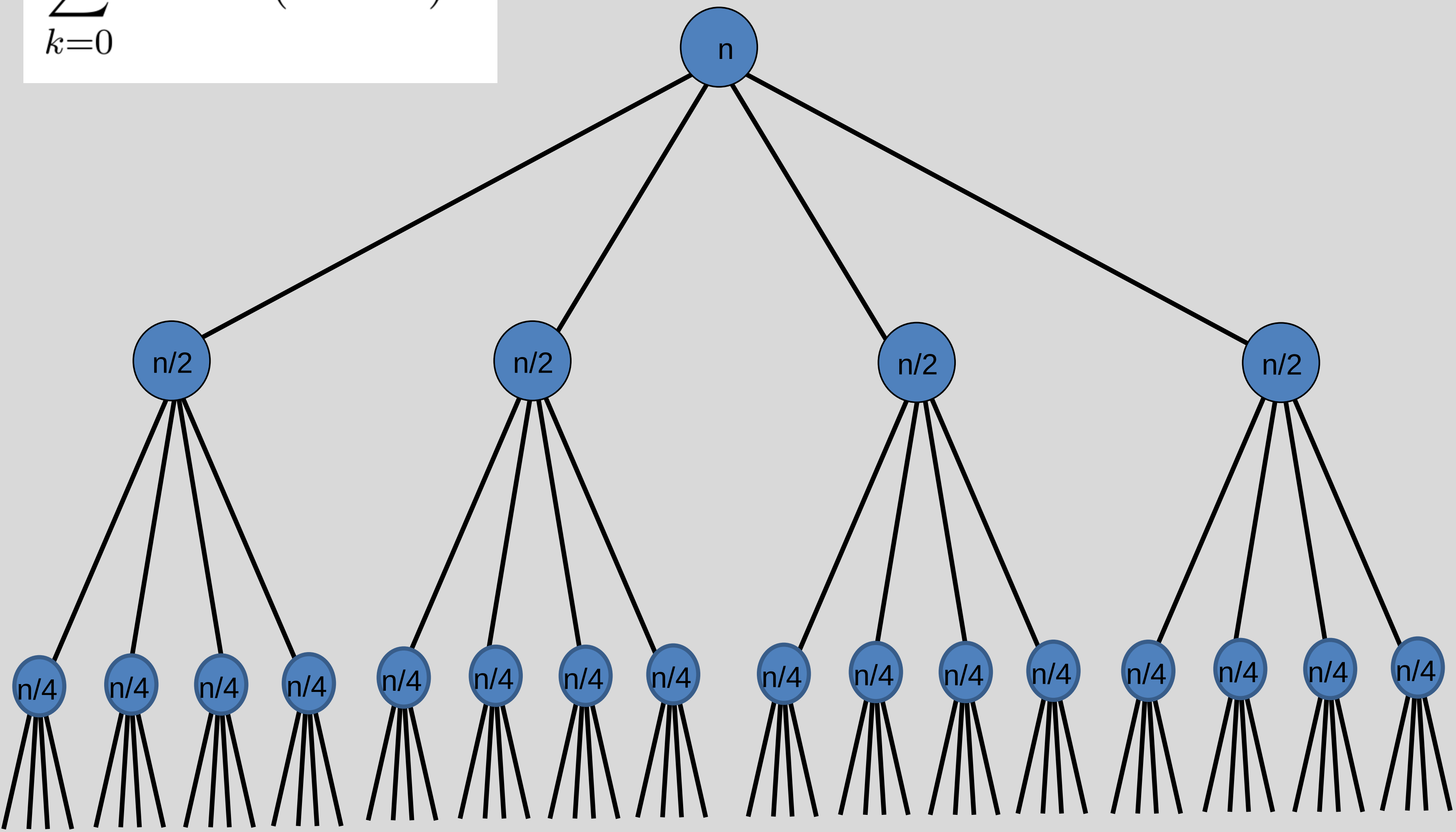
Where does this recurrence arise?

Solving the recurrence exactly

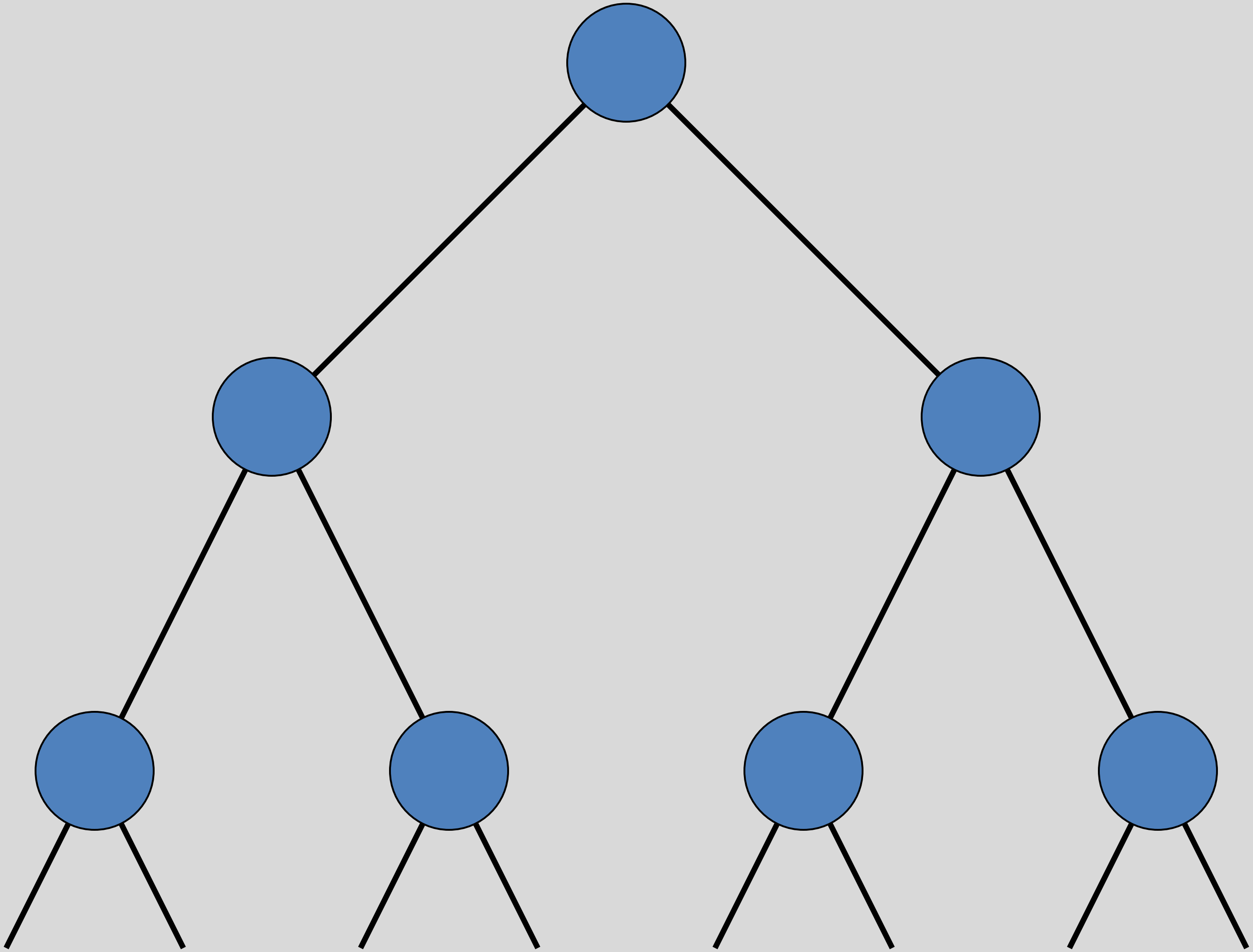
Total Work

$$\sum_{k=0}^{\log n} 2^k n = (2n-1)n$$

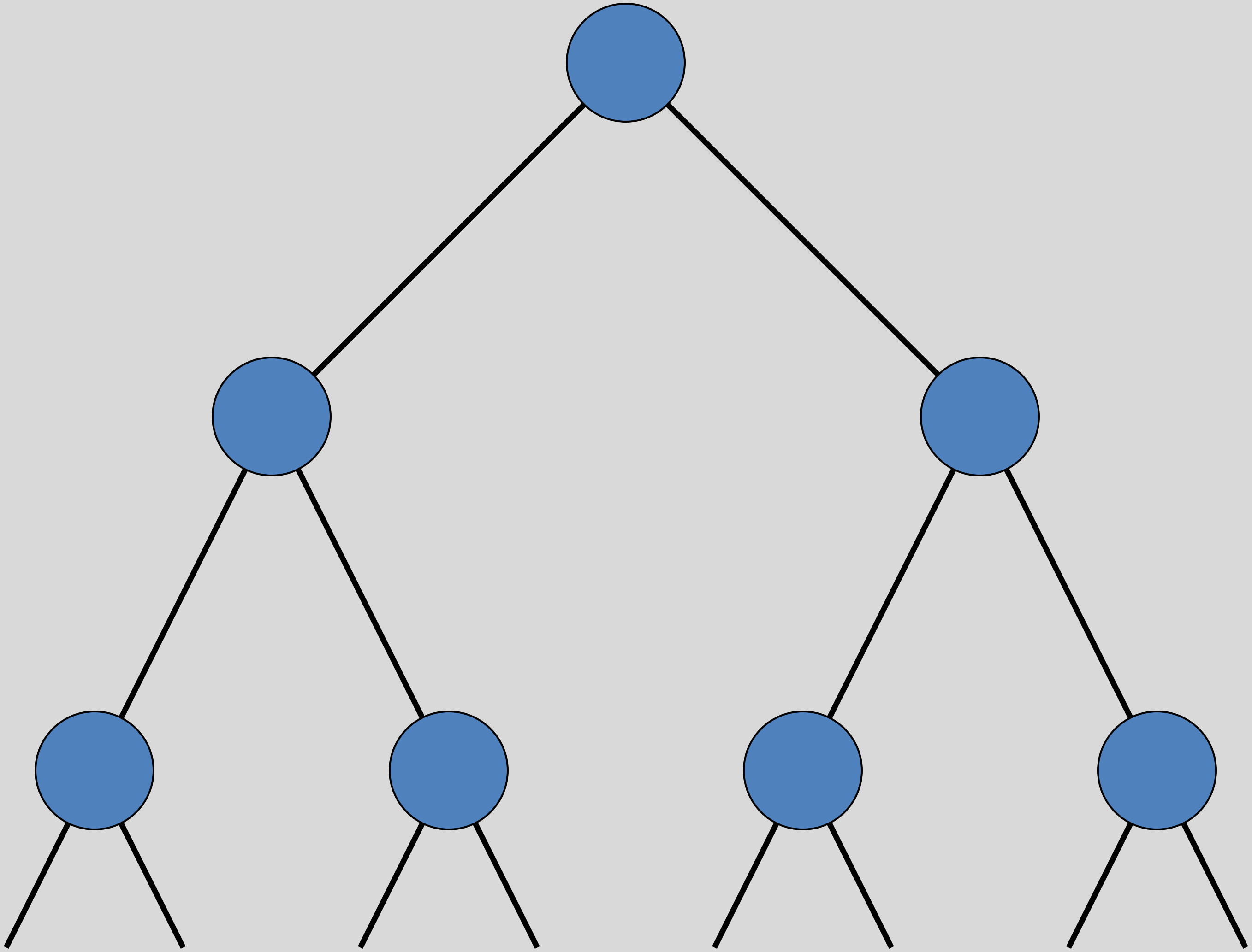
T(n) = 4T(n/2) + n



$T(n) = 2T(n/2) + n^2$



$T(n) = 2T(n/2) + n^{1/2}$



Recurrences

- Three basic behaviors
  - Dominated by initial case
  - Dominated by base case
  - All cases equal – we care about the depth

What you really need to know about recurrences

- Work per level changes geometrically with the level
- Geometrically increasing ($x > 1$)
  - The bottom level wins
- Geometrically decreasing ($x < 1$)
  - The top level wins
- Balanced ($x = 1$)
  - Equal contribution

Classify the following recurrences (Increasing, Decreasing, Balanced)

- $T(n) = n + 5T(n/8)$

- $T(n) = n + 9T(n/8)$

- $T(n) = n^2 + 4T(n/2)$

- $T(n) = n^3 + 7T(n/2)$

- $T(n) = n^{1/2} + 3T(n/4)$

Recursive Matrix Multiplication

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = ae + bf$

$s = ag + bh$

$t = ce + df$

$u = cg + dh$

A N x N matrix can be viewed as a 2 x 2 matrix with entries that are (N/2) x (N/2) matrices.

The recursive matrix multiplication algorithm recursively multiplies the (N/2) x (N/2) matrices and combines them using the equations for multiplying 2 x 2 matrices

Recursive Matrix Multiplication

- How many recursive calls are made at each level?

- How much work in combining the results?

- What is the recurrence?

What is the run time for the recursive Matrix Multiplication Algorithm?

- Recurrence:

Strassen's Algorithm

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = p_1 + p_2 – p_4 + p_6$

$s = p_4 + p_5$

$t = p_6 + p_7$

$u = p_2 - p_3 + p_5 - p_7$

Where:

$p_1 = (b – d)(f + h)$

$p_2 = (a + d)(e + h)$

$p_3 = (a – c)(e + g)$

$p_4 = (a + b)h$

$p_5 = a(g – h)$

$p_6 = d(f – e)$

$p_7 = (c + d)e$

From AHU 1974

Recurrence for Strassen's Algorithms

- $T(n) = 7\ T(n/2) + cn^2$
- What is the runtime?

$\log_2 7 = 2.8073549221$

BFPRT Recurrence

$$T(n) \leq T(3n/4) + T(n/5) + 20\, n$$

What bound do you expect?