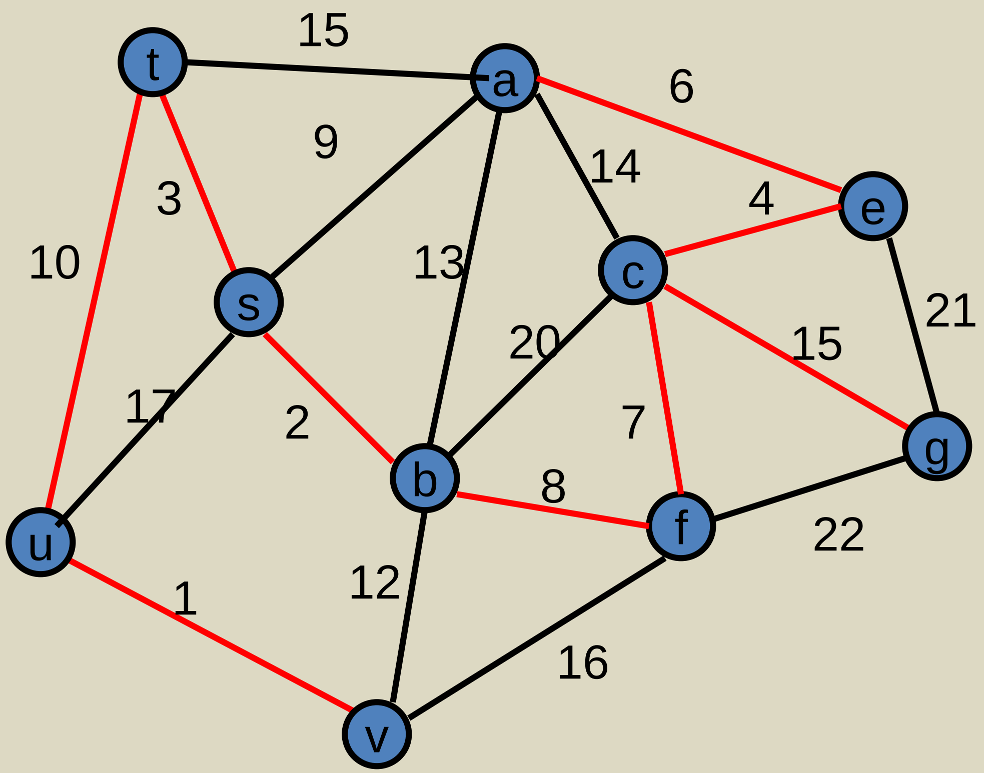
CSE 417
Algorithms and Complexity

Winter 2023
Lecture 13
Minimum Spanning Trees

Announcements

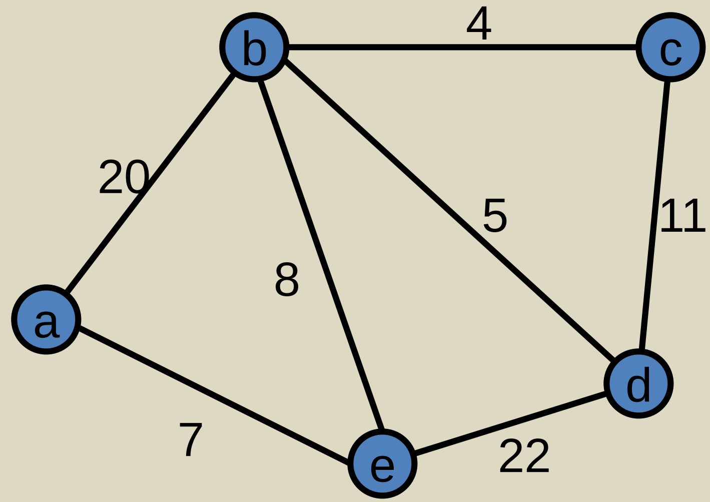
- Midterm, Wednesday, Feb 8

Minimum Spanning Tree



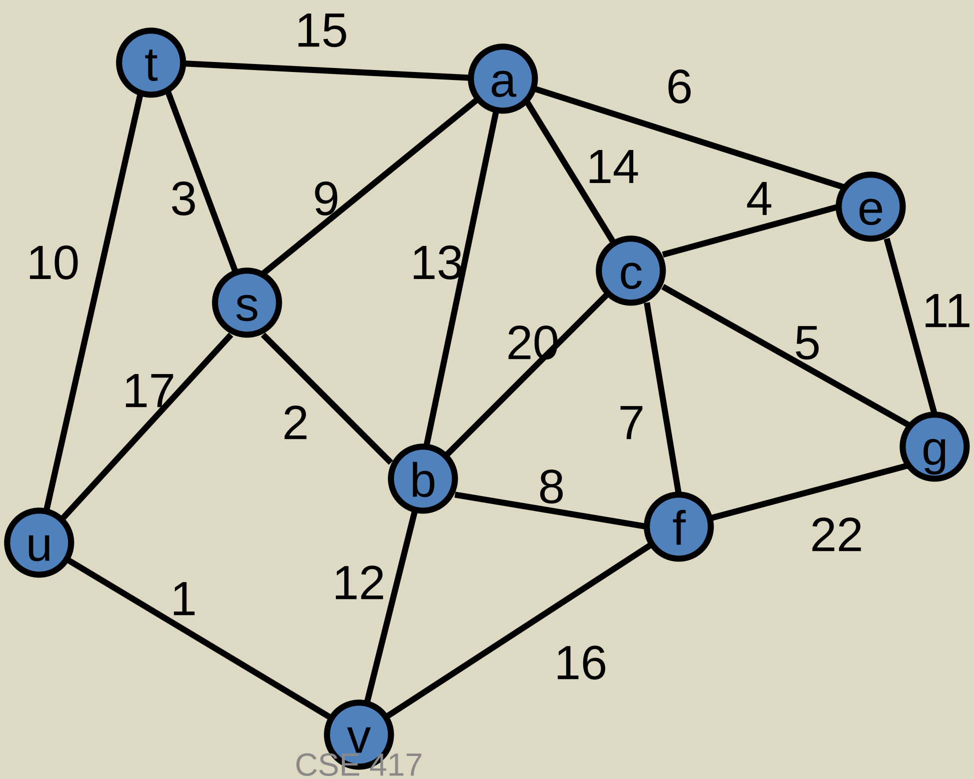
Greedy Algorithms for Minimum Spanning Tree

- Prim's Algorithm: Extend a tree by including the cheapest out going edge
- Kruskal's Algorithm: Add the cheapest edge that joins disjoint components



Greedy Algorithm 1
Prim's Algorithm

- Extend a tree by including the cheapest out going edge

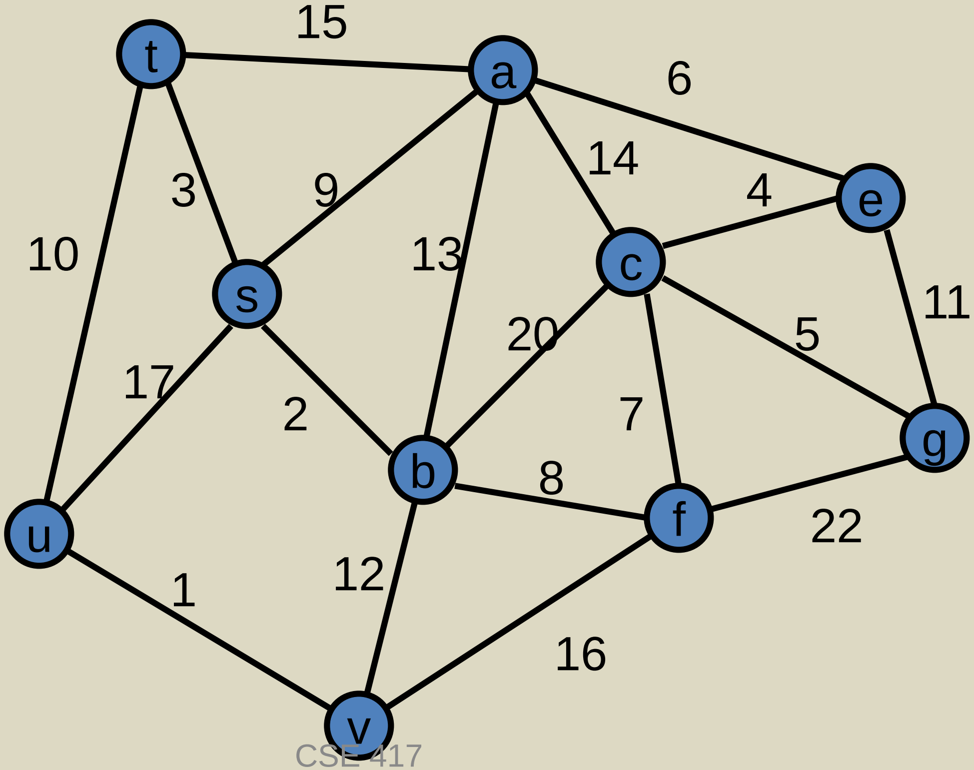


Construct the MST with Prim's algorithm starting from vertex a

Label the edges in order of insertion

Greedy Algorithm 2
Kruskal's Algorithm

- Add the cheapest edge that joins disjoint components



Construct the MST with Kruskal's algorithm

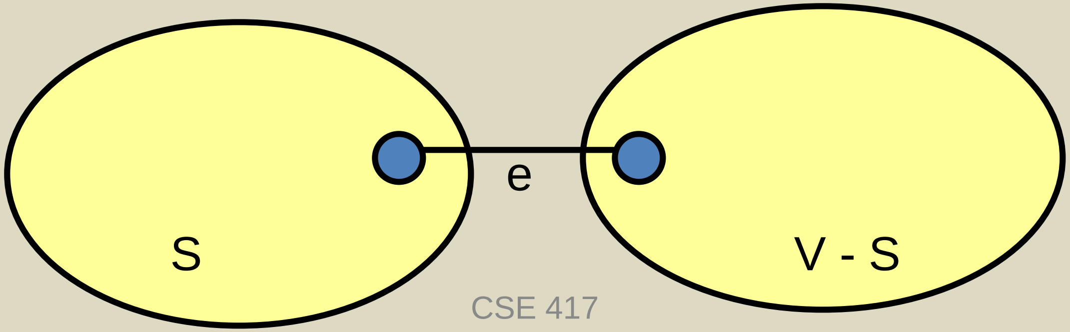
Label the edges in order of insertion

Why do the greedy algorithms work?

- For simplicity, assume all edge costs are distinct

Edge inclusion lemma

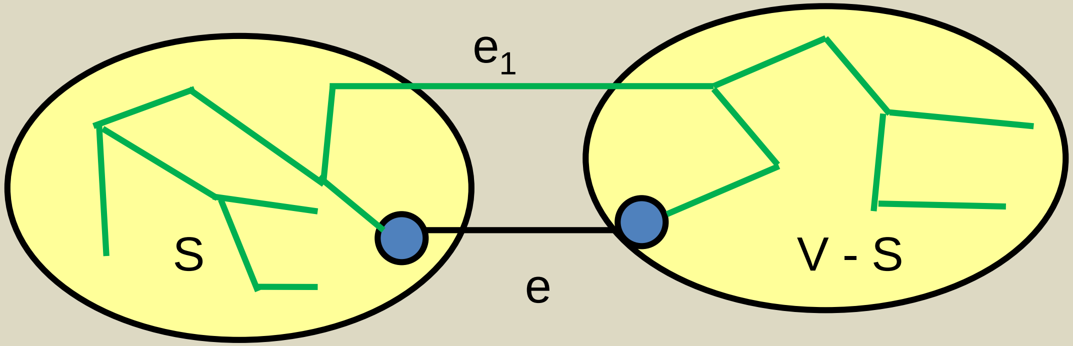
- Let S be a subset of V, and suppose e = (u, v) is the minimum cost edge of E, with u in S and v in V-S
- e is in every minimum spanning tree of G
  - Or equivalently, if e is not in T, then T is not a minimum spanning tree



e is the minimum cost edge between S and V-S

Proof

- Suppose T is a spanning tree that does not contain e
- Add e to T, this creates a cycle
- The cycle must have some edge $e_1 = (u_1, v_1)$ with u_1 in S and v_1 in V-S



- $T_1 = T - \{e_1\} + \{e\}$ is a spanning tree with lower cost
- Hence, T is not a minimum spanning tree

Optimality Proofs

- Prim’s Algorithm computes a MST
- Kruskal’s Algorithm computes a MST

- Show that when an edge is added to the MST by Prim or Kruskal, the edge is the minimum cost edge between S and V-S for some set S.

Prim's Algorithm

S = { };   T = { };

while S != V

    choose the minimum cost edge
    e = (u,v), with u in S, and v in V-S

    add e to T

    add v to S

Prove Prim's algorithm computes an MST

- Show an edge e is in the MST when it is added to T

Kruskal's Algorithm

Let $C = \{\{v_1\}, \{v_2\}, . . ., \{v_n\}\}$; $T = \{ \}$

while $|C| > 1$

> Let $e = (u, v)$ with u in C_i and v in C_j be the minimum cost edge joining distinct sets in C
>
> Replace C_i and C_j by $C_i \cup C_j$
>
> Add e to T

Prove Kruskal’s algorithm computes an MST

- Show an edge e is in the MST when it is added to T

MST Implementation and runtime

- Prim's Algorithm
  - Implementation, runtime: just like Dijkstra's algorithm
  - Use a heap, runtime $O(m \log n)$
- Kruskal's Algorithm
  - Sorting edges by cost: $O(m \log n)$
  - Managing connected components uses the Union-Find data structure
    - Amazing, pointer based data structure
    - Very interesting mathematical result

Disjoint Set ADT

- Data: set of pairwise **disjoint sets**.
- Required operations
  - **Union** – merge two sets to create their union
  - **Find** – determine which set an item appears in

- Check Find(v) $\neq$ Find(w) to determine if (v,w) joins separate components
- Do Union(v,w) to merge sets

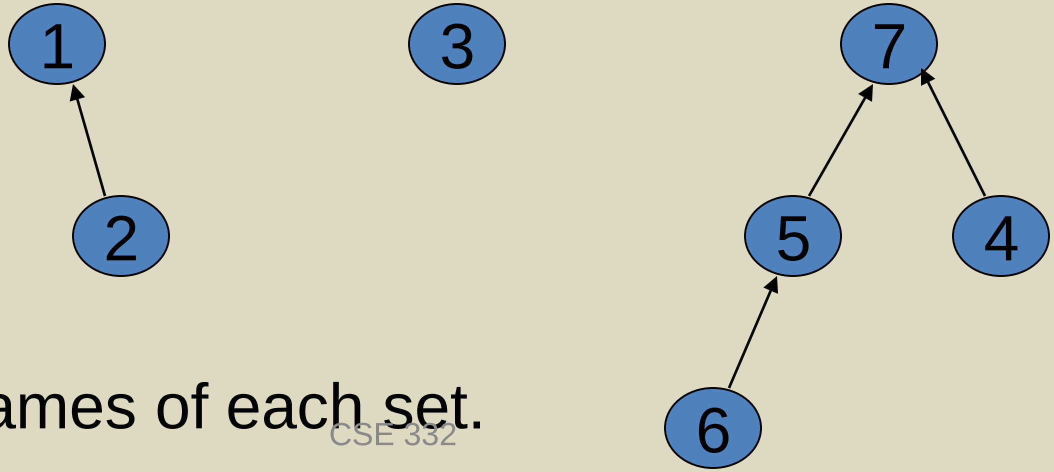
Up-Tree for DS Union/Find

Observation: we will only traverse these trees upward from any given node to find the root.

Idea: *reverse* the pointers (make them point up from child to parent). The result is an **up-tree**.

Initial state

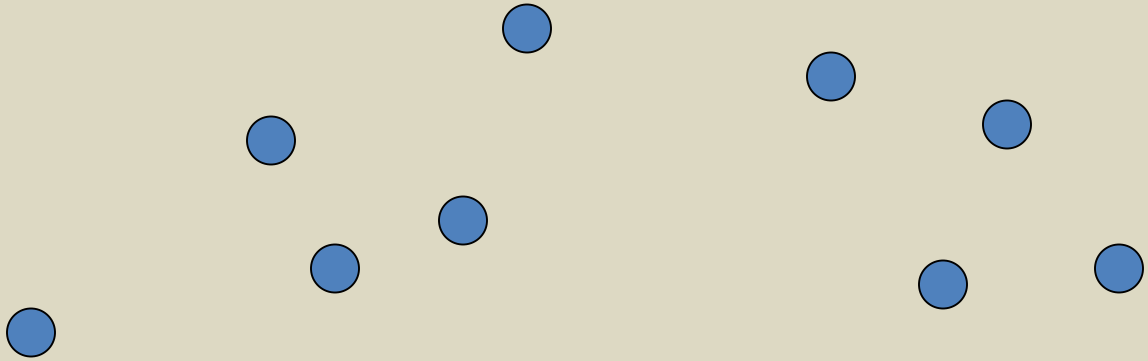
Intermediate state



Roots are the names of each set.

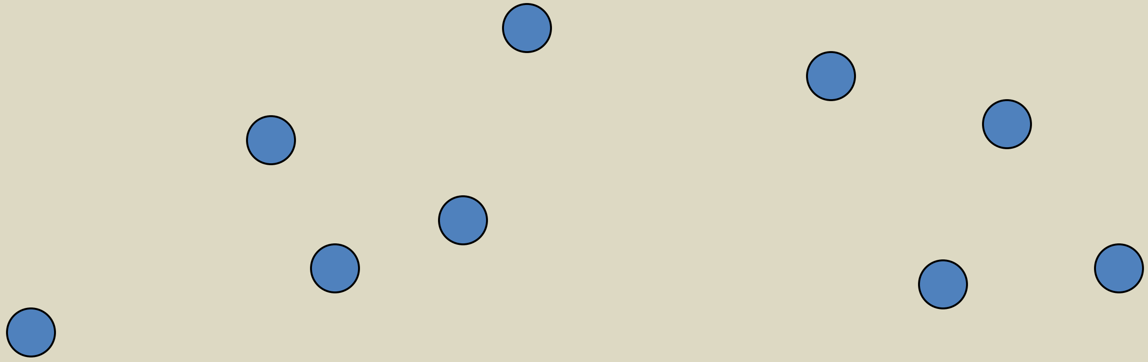
Application: Clustering

- Given a collection of points in an r-dimensional space and an integer K, divide the points into K sets that are closest together

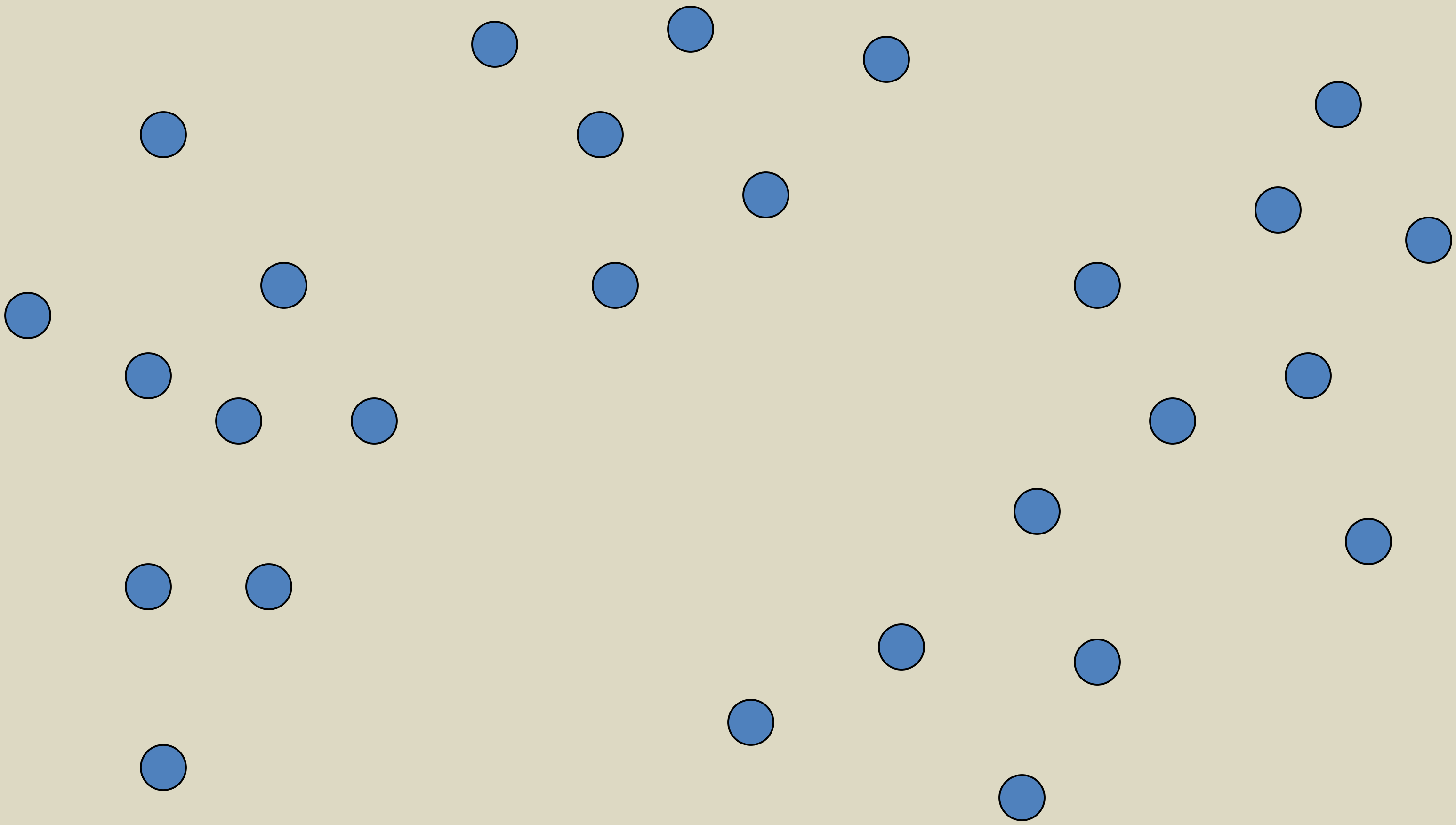


Distance clustering

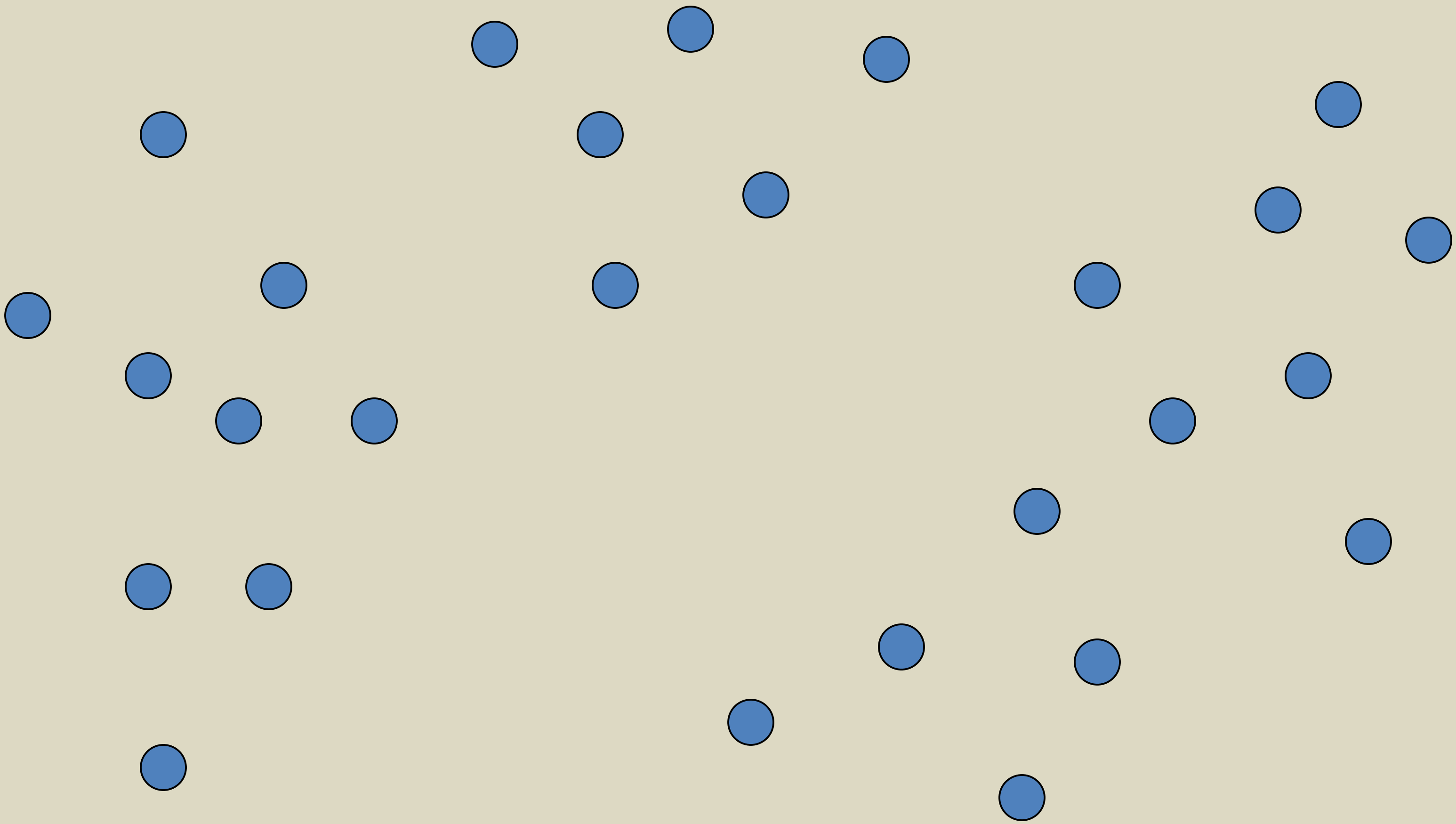
- Divide the data set into K subsets to maximize the distance between any pair of sets
  - $\text{dist}(S_1, S_2) = \min \{\text{dist}(x, y) \mid x \text{ in } S_1, y \text{ in } S_2\}$



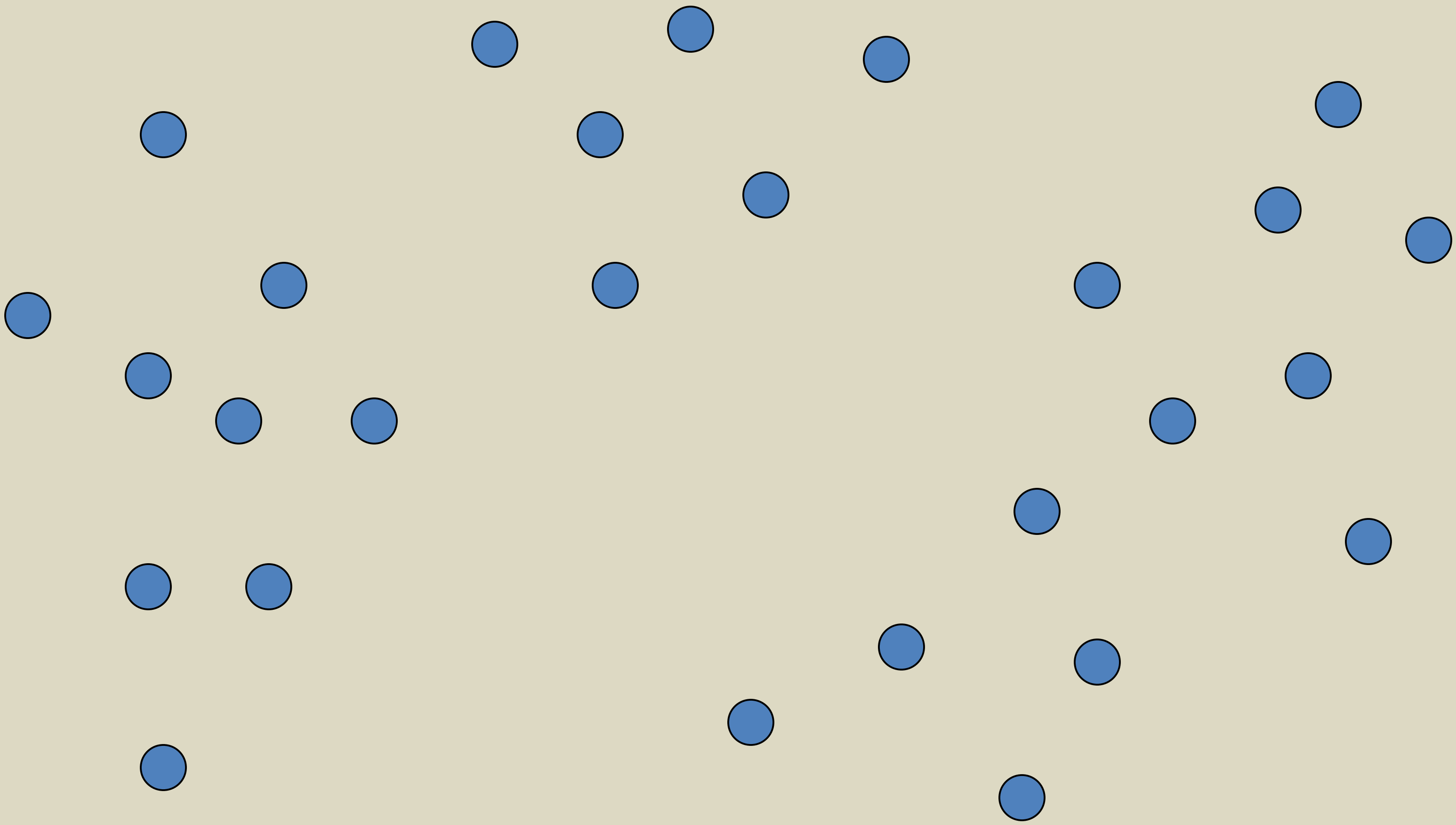
Divide into 2 clusters



Divide into 3 clusters



Divide into 4 clusters



Distance Clustering Algorithm

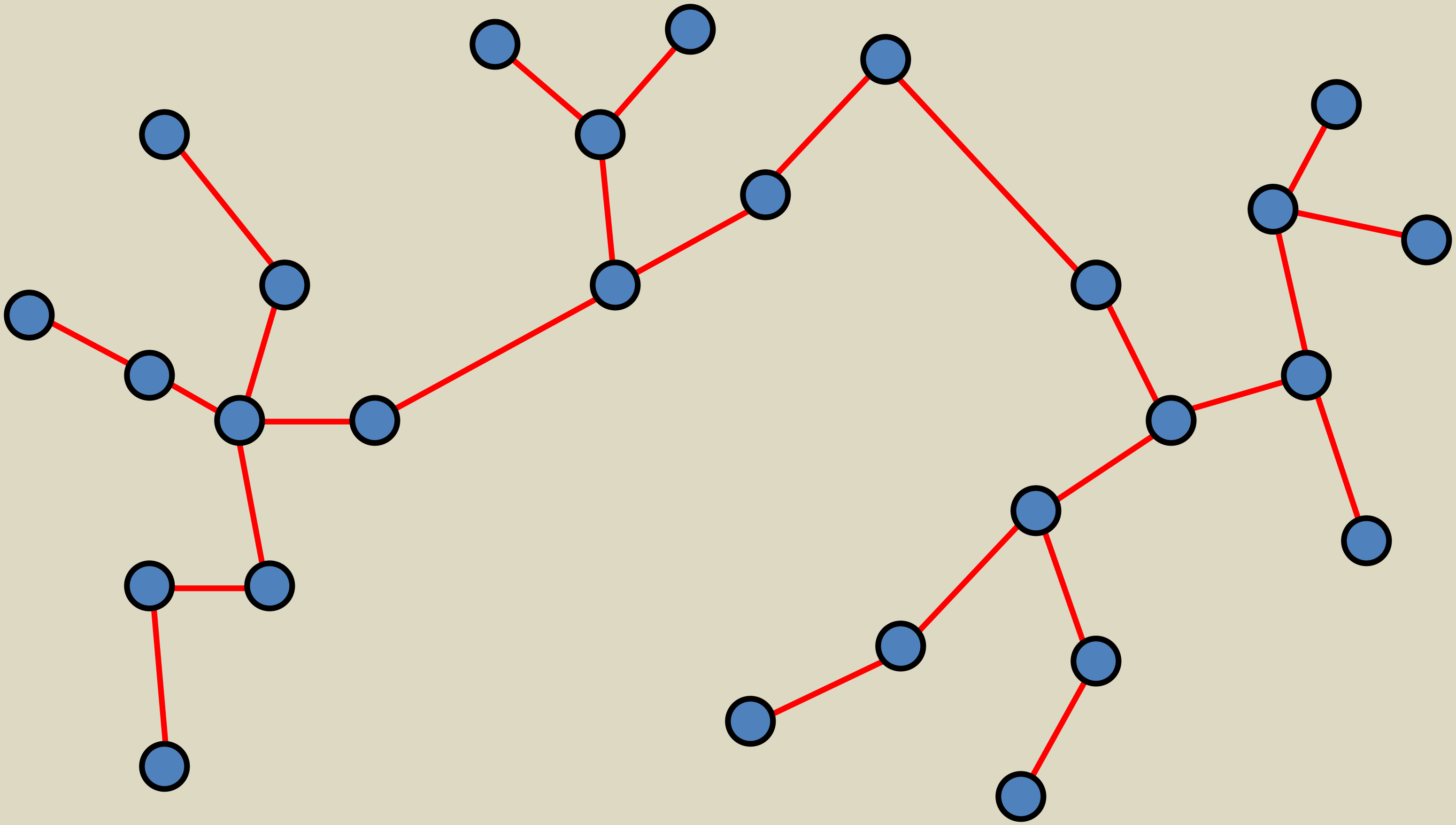
Let $C = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$; $T = \{\ \}$

while $|C| > K$

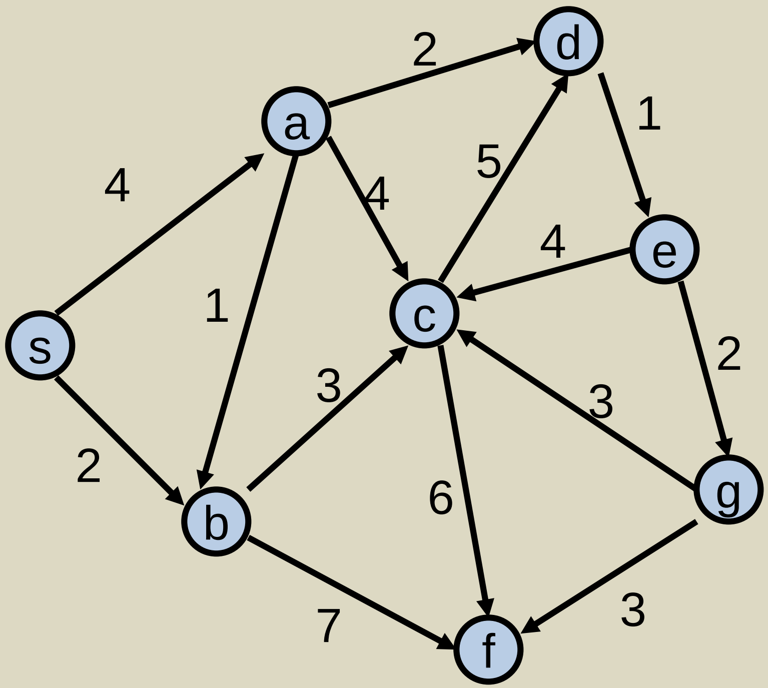
Let $e = (u, v)$ with u in C_i and v in C_j be the minimum cost edge joining distinct sets in C

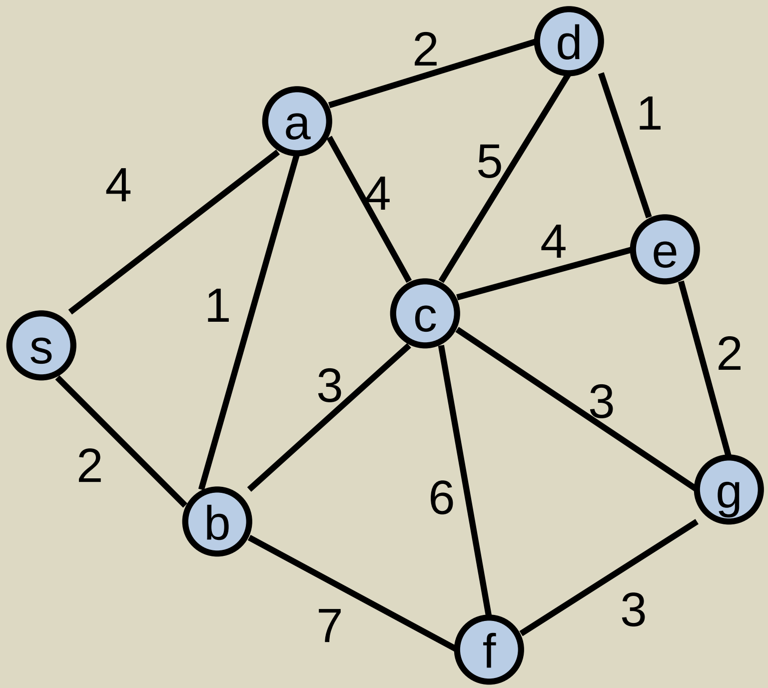
Replace C_i and C_j by $C_i \cup C_j$

K-clustering



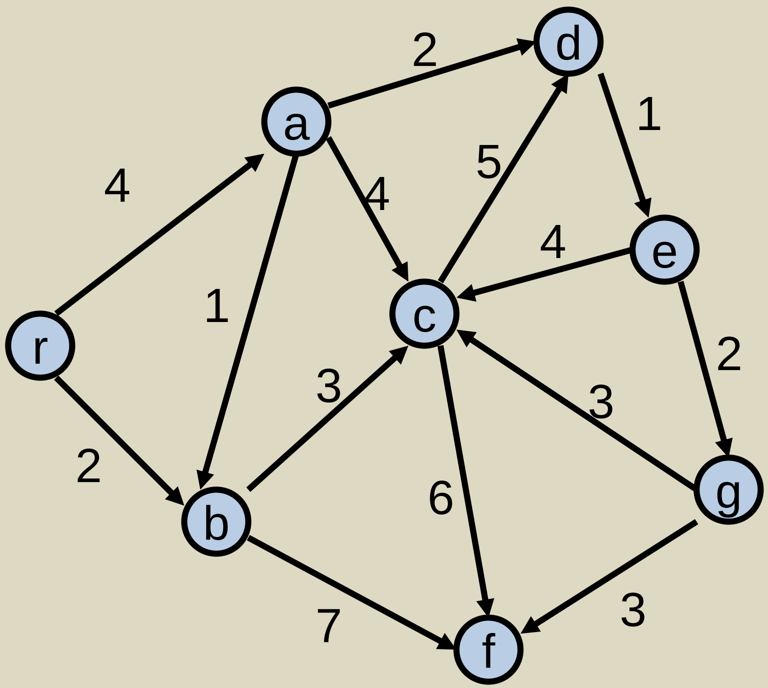
Shortest paths in directed graphs vs undirected graphs



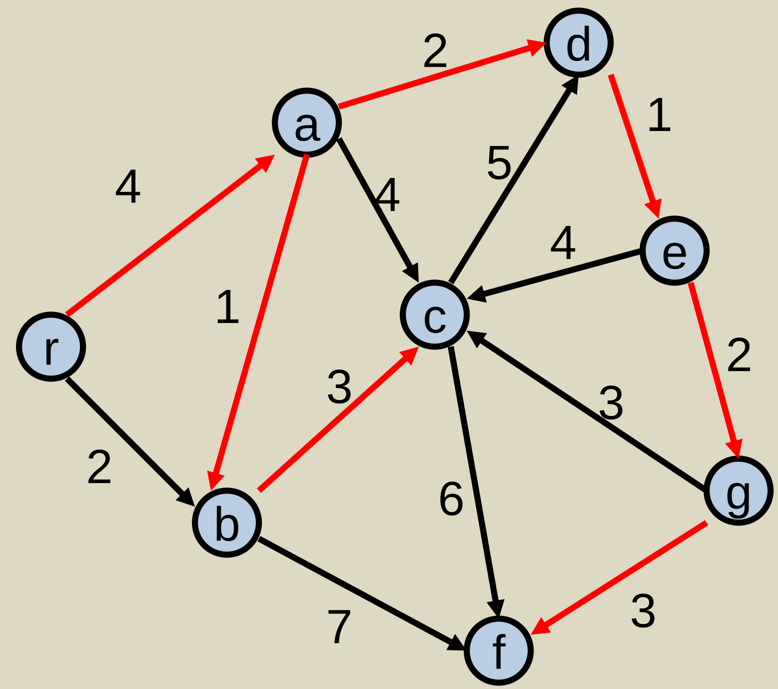


What about the minimum spanning tree of a directed graph?

- Must specify the root r
- Branching: Out tree with root r

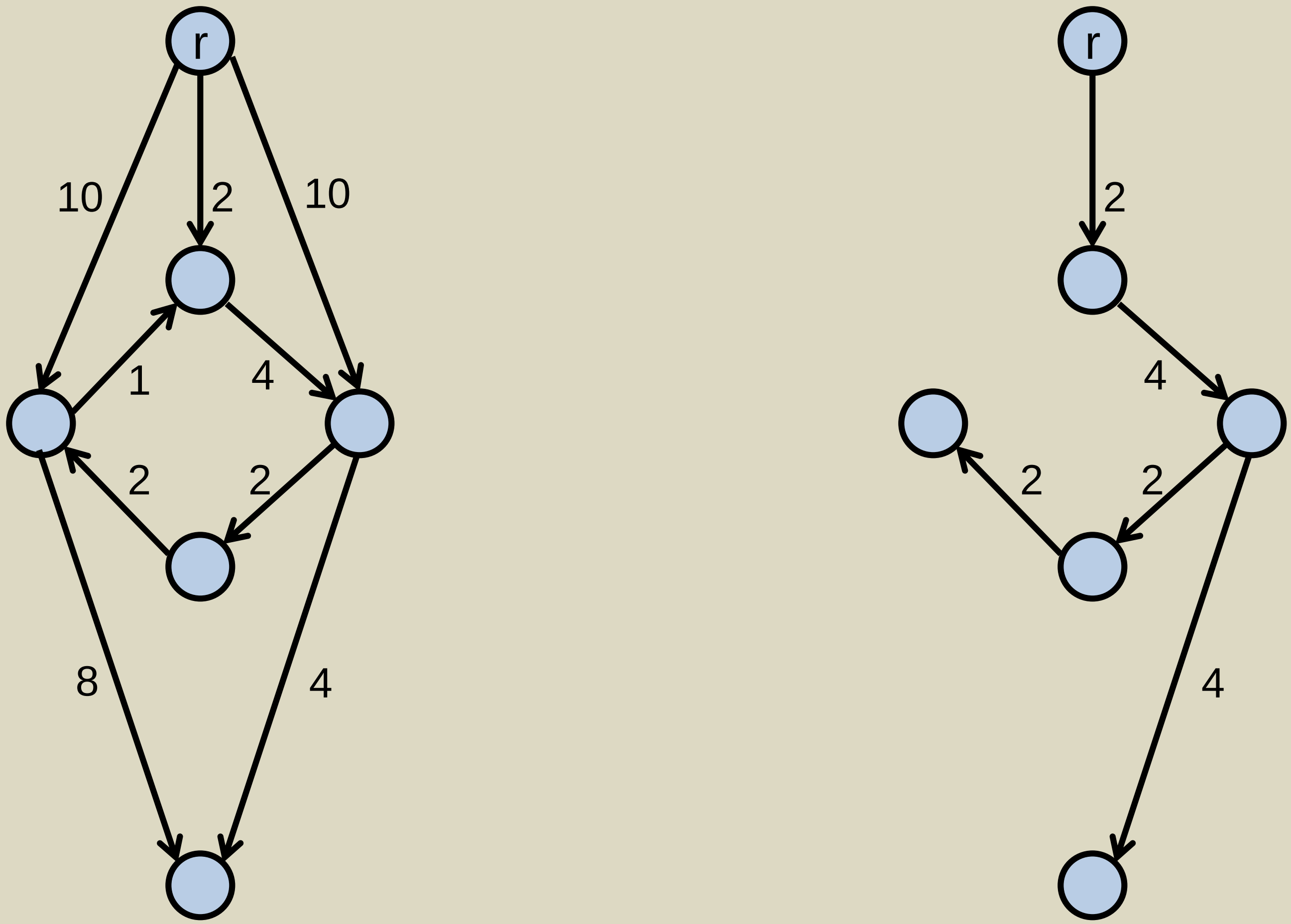


Assume all vertices reachable from r



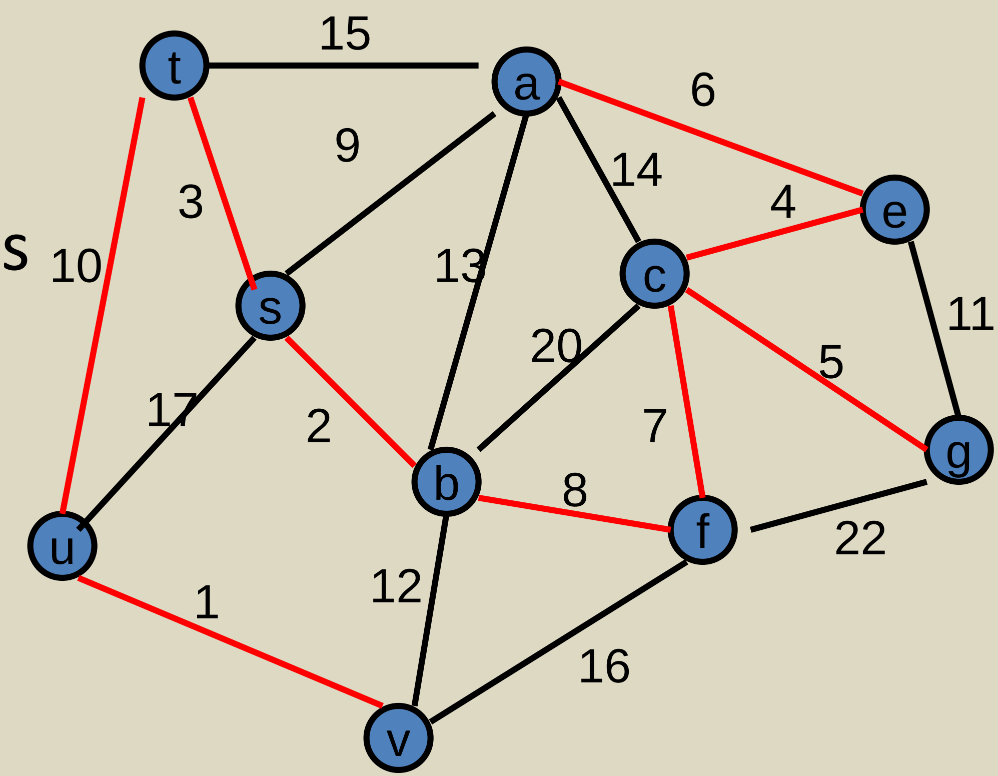
Also called an arborescence

Finding a minimum branching



Another MST Algorithm

- Choose minimum cost edge into each vertex
- Merge into components
- Repeat until done



Idea for branching algorithm

- Select minimum cost edge going into each vertex
- If graph is a branching then done
- Otherwise collapse cycles and repeat