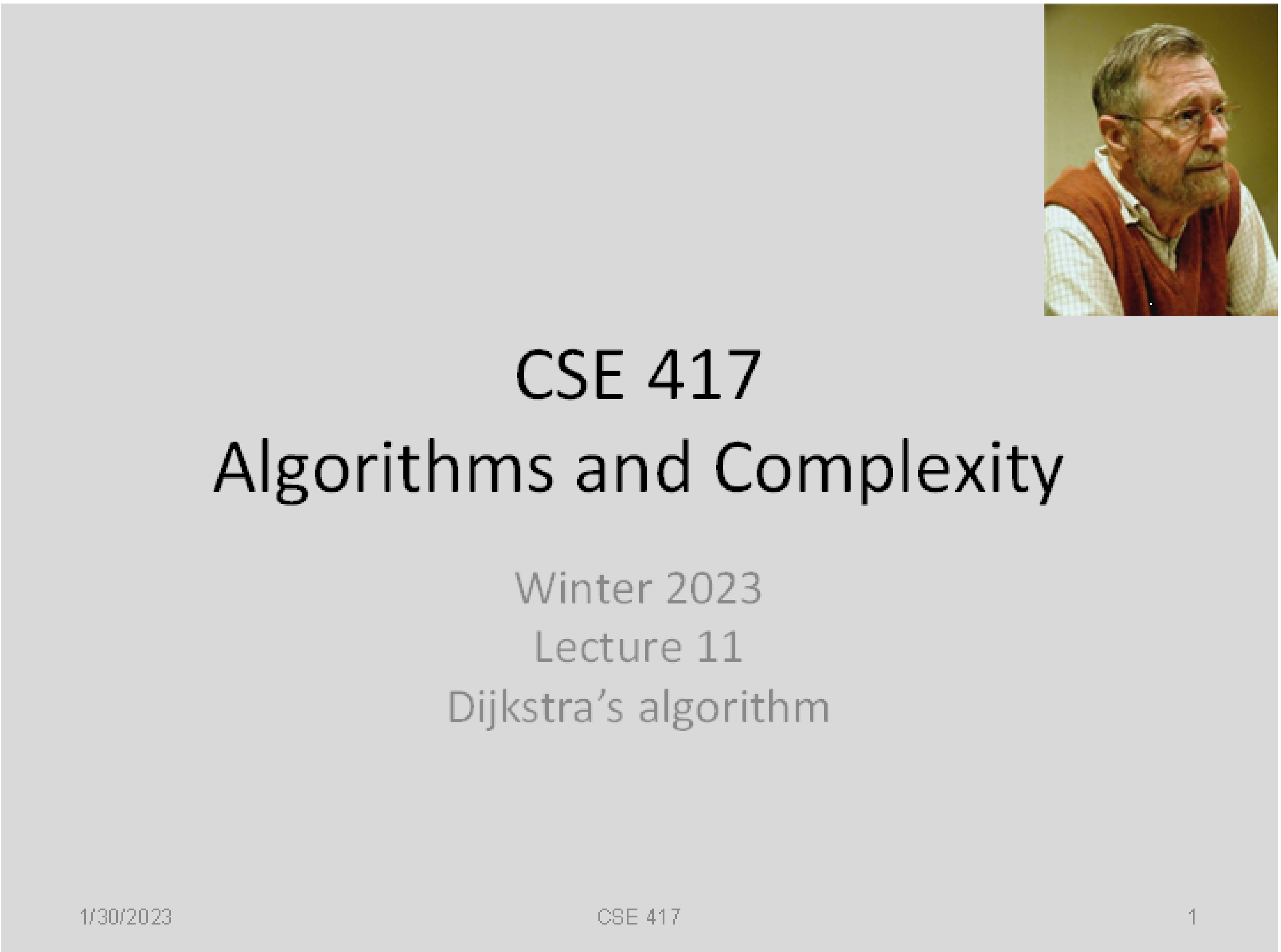
Lecture11


CSE 417
Algorithms and Complexity
Winter 2023
Lecture 11
Dijkstra's algorithm
1/30/2023
CSE 417
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Upcoming lectures

- Topics
  - Dijkstra's Algorithm (Section 4.4)
  - Wednesday: Minimum Spanning Trees
- Reading
  - 4.4, 4.5, 4.7, ~~4.8~~ 4.9

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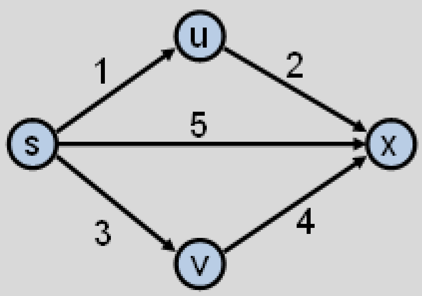
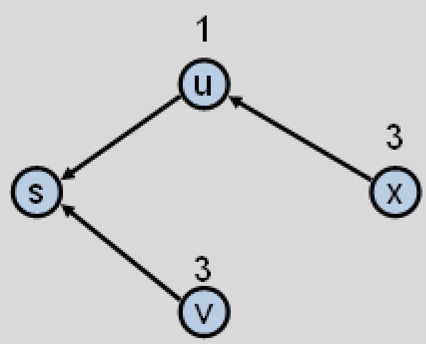
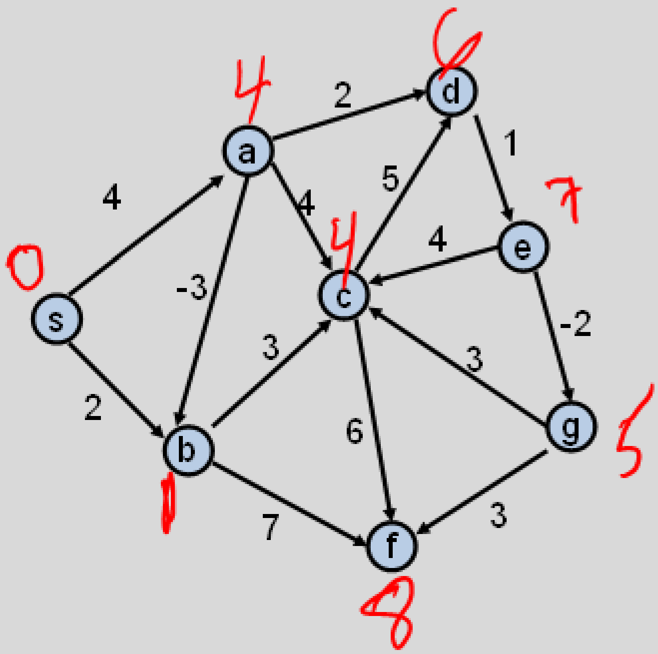
Single Source Shortest Path Problem

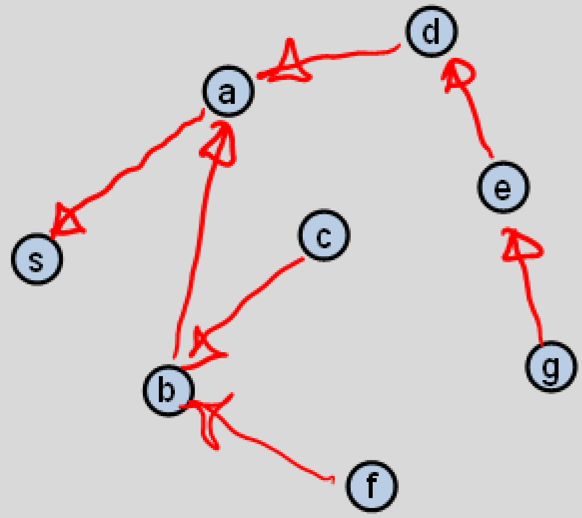
- Given a graph and a start vertex s
  - Determine distance of every vertex from s
  - Identify shortest paths to each vertex
    - Express concisely as a “shortest paths tree”
    - Each vertex has a pointer to a predecessor on shortest path





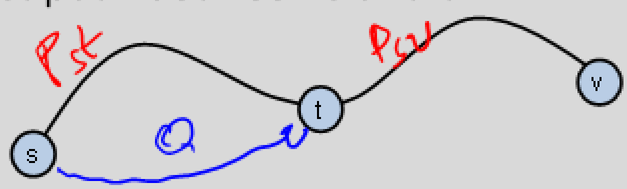
Construct Shortest Path Tree from s





Warmup

- If P is a shortest path from s to v, and if t is on the path P, the segment from s to t is a shortest path between s and t



- WHY?

P – s.p. from s to v

$P = P_{st} + P_{tv}$

claim P_{st} s.p from s to t

Suppose P_{st} not s.p. from s to t

$|Q| < |P_{st}|$

$Q + P_{tv}$

$|Q + P_{tv}| < |P_{st}| + |P_{tv}| = |P|$

Assume all edges have non-negative cost

Dijkstra's Algorithm

S = { }; d[s] = 0; d[v] = infinity for v != s

While S != V

Choose v in V-S with minimum d[v]

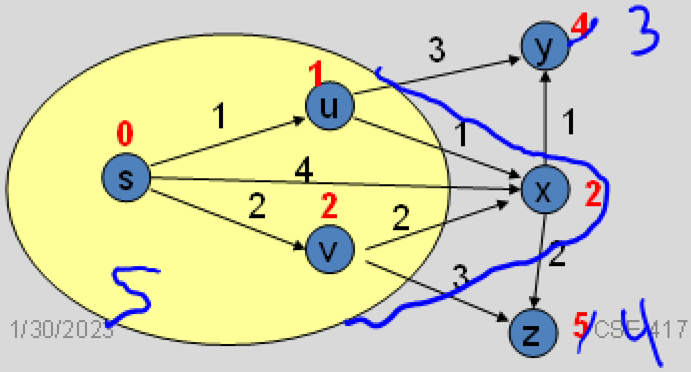
Add v to S

For each w in the neighborhood of v

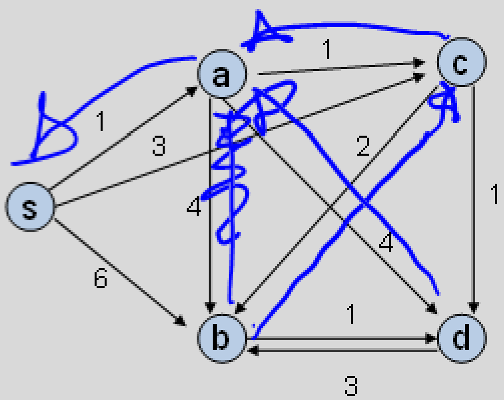
d[w] = min(d[w], d[v] + c(v, w))

O(log n)

update prev



Simulate Dijkstra's algorithm (starting from s) on the graph



Round	Vertex Added	s	a	b	c	d
		0	∞	∞	∞	∞
1	s	0	1	6	3	∞
2	a	0	1	5	2	5
3	c	0	1	4	2	3
4	d	0	1	4	2	3
5	b	0	1	4	2	3

Who was Dijkstra?

- What were his major contributions?

http://www.cs.utexas.edu/users/EWD/

- Edsger Wybe Dijkstra was one of the most influential members of computing science's founding generation. Among the domains in which his scientific contributions are fundamental are
  - algorithm design
  - programming languages
  - program design
  - operating systems
  - distributed processing
  - formal specification and verification
  - design of mathematical arguments

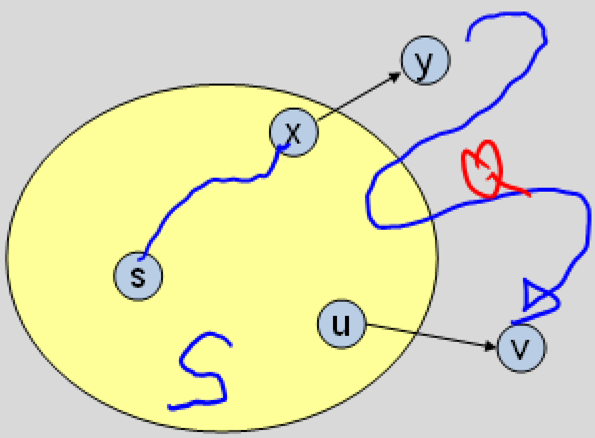
Dijkstra's Algorithm as a greedy algorithm

- Elements committed to the solution by order of minimum distance

Correctness Proof

$d[y] \geq d[v]$

- Elements in S have the correct label
- Key to proof: when v is added to S, it has the correct distance label.



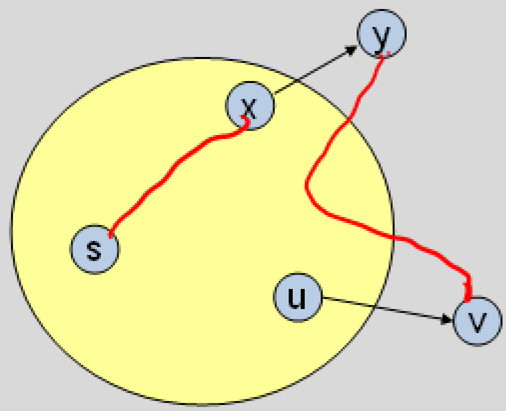
look at a path from s to v that does not go through u

$|Q| \geq 0$

$O(m \log n)$

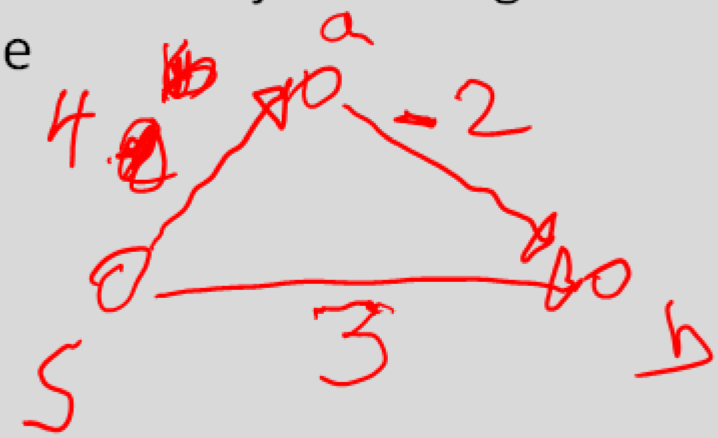
Proof

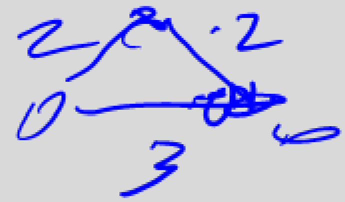
- Let v be a vertex in V-S with minimum d[v]
- Let P_v be a path of length d[v], with an edge (u,v)
- Let P be some other path to v. Suppose P first leaves S on the edge (x, y)
  - $P = P_{sx} + c(x,y) + P_{yv}$
  - $Len(P_{sx}) + c(x,y) >= d[y]$
  - $Len(P_{yv}) >= 0$
  - $Len(P) >= d[y] + 0 >= d[v]$



Negative Cost Edges

- Draw a small example a negative cost edge and show that Dijkstra's algorithm fails on this example





Dijkstra Implementation

```
S = { };    d[s] = 0;     d[v] = infinity for v != s
While S != V
        Choose v in V-S with minimum d[v]
        Add v to S
        For each  w in the neighborhood of v
                d[w] = min(d[w], d[v] + c(v, w))
```

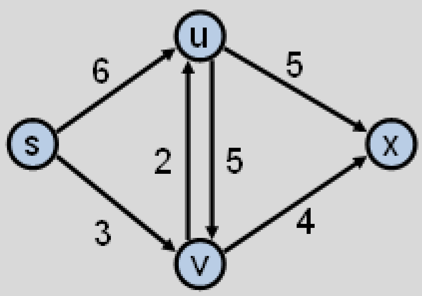
- Basic implementation requires Heap for tracking the distance values
- Run time O(m log n)

$O(n^2)$ Implementation for Dense Graphs

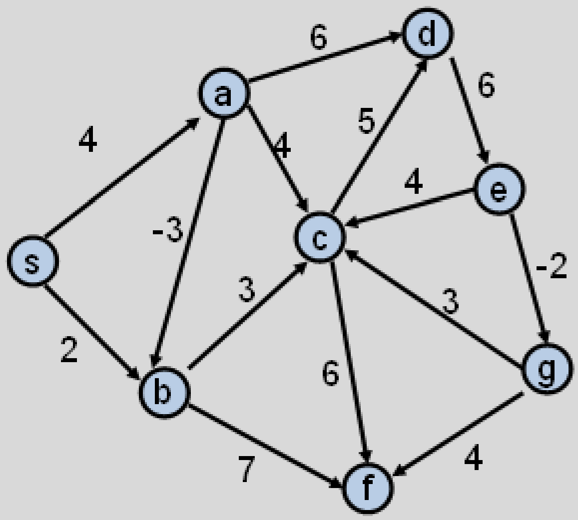
```
FOR i := 1 TO n
       d[i] := Infinity;  visited[i] := FALSE;
d[s] := 0;

FOR i := 1 TO n
       v := -1;  dMin := Infinity;
       FOR j := 1 TO n
              IF visited[j] = FALSE AND d[j] < dMin
                     v := j; dMin := d[j];
       IF v = -1
              RETURN;
       visited[v] := TRUE;

       FOR j := 1 TO n
              IF d[v] + len[v, j] < d[j]
                     d[j] := d[v] + len[v, j];
                     prev[j] := v;
```

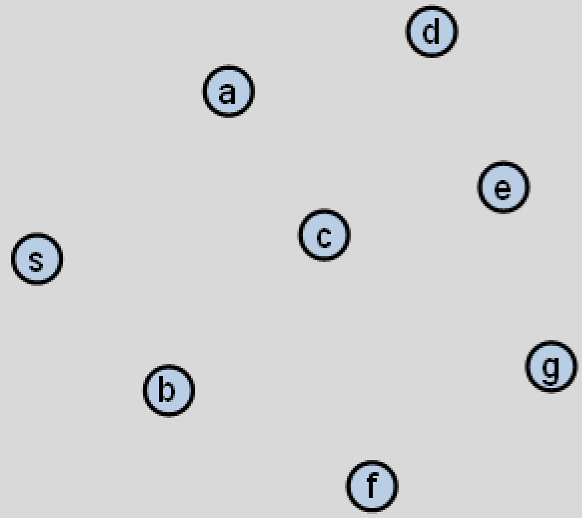
Bottleneck Shortest Path

- Define the bottleneck distance for a path to be the maximum cost edge along the path



Compute the bottleneck shortest paths





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How do you adapt Dijkstra's algorithm to handle bottleneck distances

- Does the correctness proof still apply?