CSE 417
Algorithms and Complexity

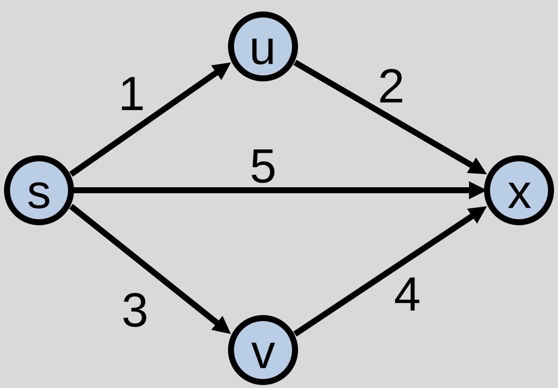
Winter 2023
Lecture 11
Dijkstra’s algorithm

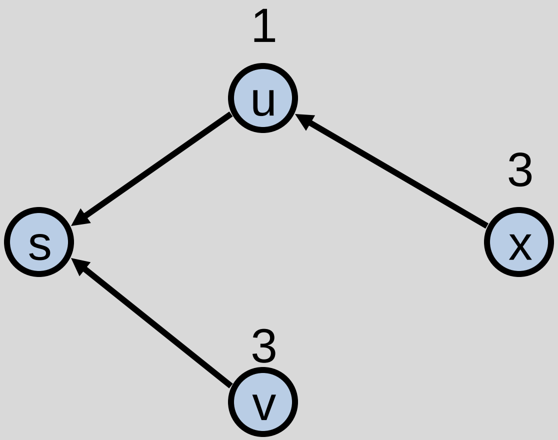
Upcoming lectures

- Topics
  - Dijkstra's Algorithm (Section 4.4)
  - Wednesday: Minimum Spanning Trees
- Reading
  - 4.4, 4.5, 4.7, 4.8

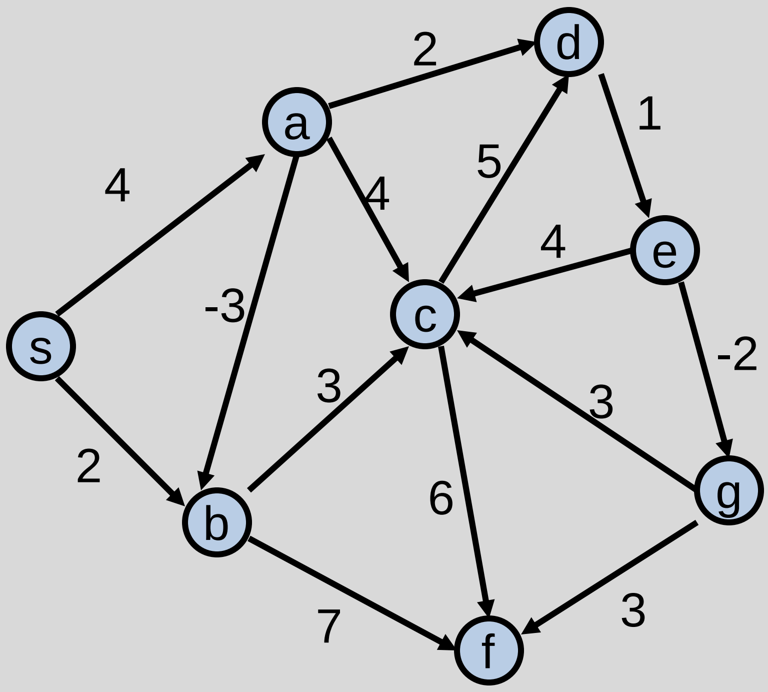
Single Source Shortest Path Problem

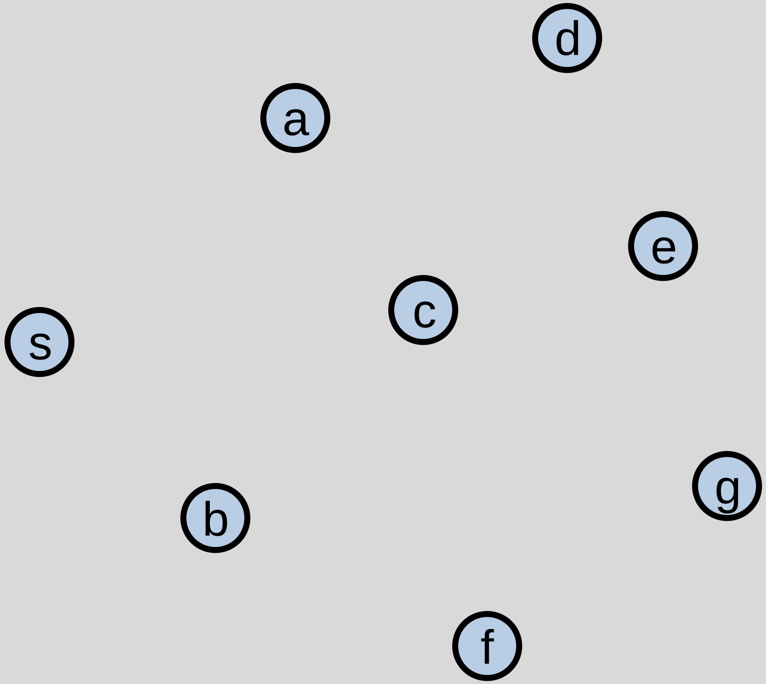
- Given a graph and a start vertex s
  - Determine distance of every vertex from s
  - Identify shortest paths to each vertex
    - Express concisely as a "shortest paths tree"
    - Each vertex has a pointer to a predecessor on shortest path





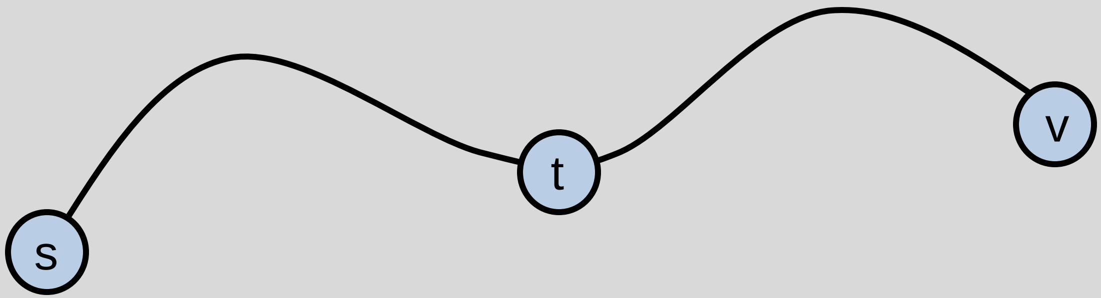
Construct Shortest Path Tree from s





Warmup

- If P is a shortest path from s to v, and if t is on the path P, the segment from s to t is a shortest path between s and t



- WHY?

Assume all edges have non-negative cost

Dijkstra's Algorithm

S = { }; d[s] = 0; d[v] = infinity for v != s

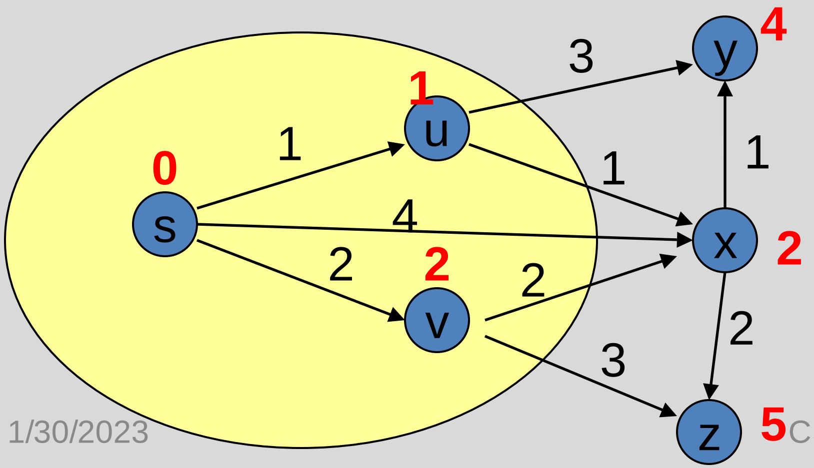
While S != V

  Choose v in V-S with minimum d[v]

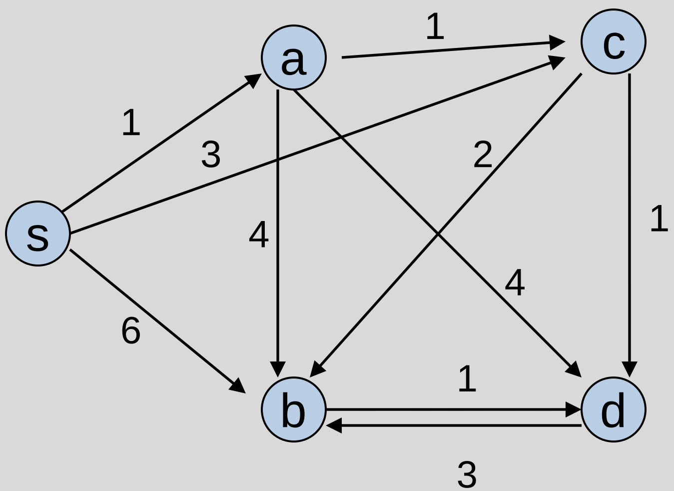
  Add v to S

  For each w in the neighborhood of v

    d[w] = min(d[w], d[v] + c(v, w))



Simulate Dijkstra's algorithm (starting from s) on the graph



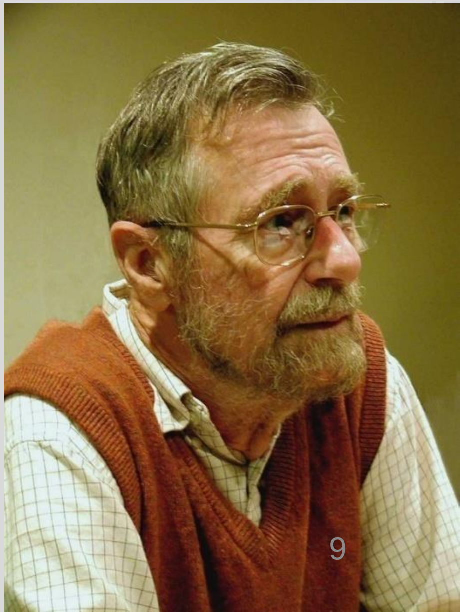
Round	Vertex Added	s	a	b	c	d
1						
2						
3						
4						
5						

Who was Dijkstra?

- What were his major contributions?

http://www.cs.utexas.edu/users/EWD/

- Edsger Wybe Dijkstra was one of the most influential members of computing science's founding generation. Among the domains in which his scientific contributions are fundamental are
  - algorithm design
  - programming languages
  - program design
  - operating systems
  - distributed processing
  - formal specification and verification
  - design of mathematical arguments

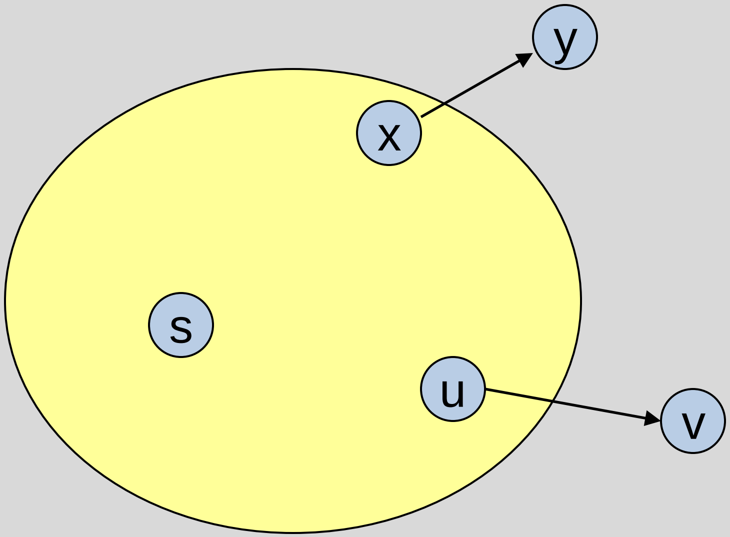


Dijkstra’s Algorithm as a greedy algorithm

- Elements committed to the solution by order of minimum distance

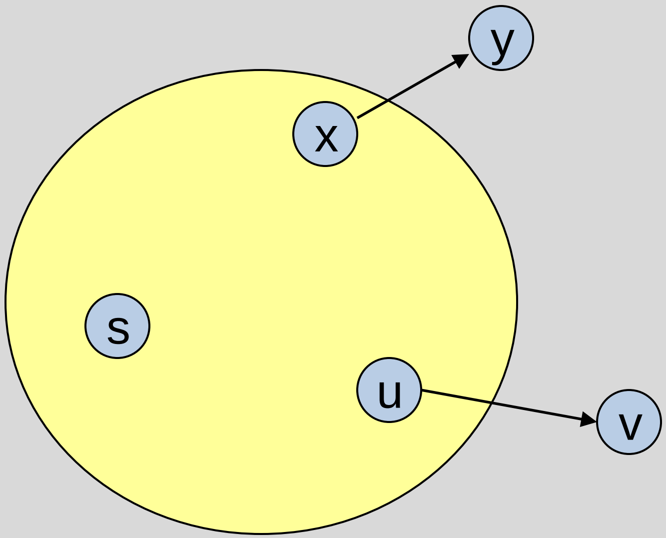
Correctness Proof

- Elements in S have the correct label
- Key to proof: when v is added to S, it has the correct distance label.



Proof

- Let v be a vertex in V-S with minimum d[v]
- Let P_v be a path of length d[v], with an edge (u,v)
- Let P be some other path to v. Suppose P first leaves S on the edge (x, y)
  - $P = P_{sx} + c(x,y) + P_{yv}$
  - $Len(P_{sx}) + c(x,y) >= d[y]$
  - $Len(P_{yv}) >= 0$
  - $Len(P) >= d[y] + 0 >= d[v]$



Negative Cost Edges

- Draw a small example a negative cost edge and show that Dijkstra's algorithm fails on this example

Dijkstra Implementation

```
S = { };    d[s] = 0;     d[v] = infinity for v != s
While S != V
        Choose v in V-S with minimum d[v]
        Add v to S
        For each  w in the neighborhood of v
                d[w] = min(d[w], d[v] + c(v, w))
```

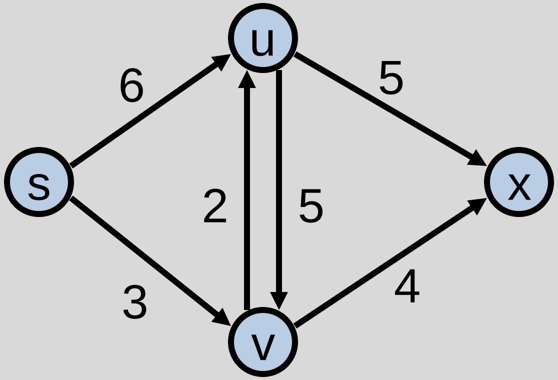
- Basic implementation requires Heap for tracking the distance values
- Run time O(m log n)

O(n^2) Implementation for Dense Graphs

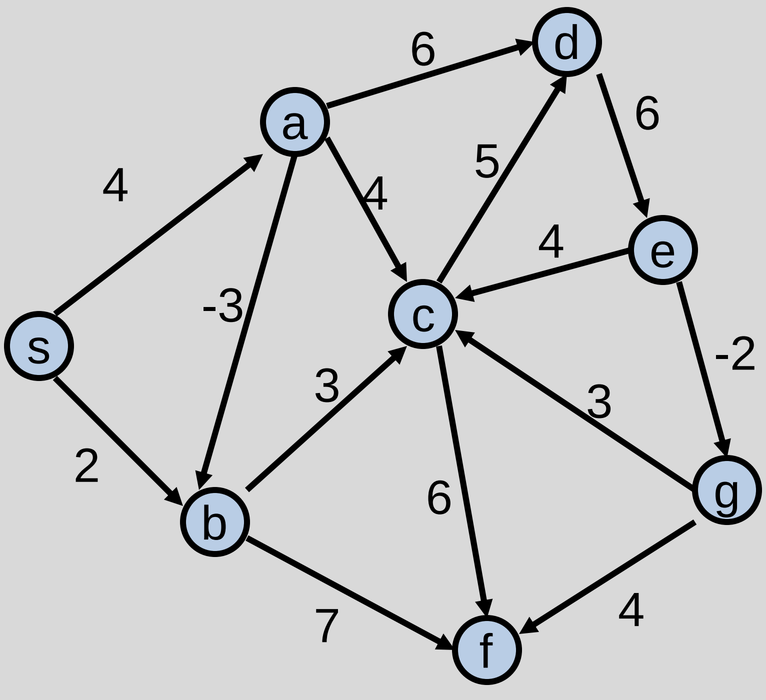
```
FOR i := 1 TO n
      d[i] := Infinity;  visited[i] := FALSE;
d[s] := 0;

FOR i := 1 TO n
      v := -1;  dMin := Infinity;
      FOR j := 1 TO n
            IF visited[j] = FALSE AND d[j] < dMin
                  v := j; dMin := d[j];
      IF v = -1
            RETURN;
      visited[v] := TRUE;

      FOR j := 1 TO n
            IF d[v] + len[v, j] < d[j]
                  d[j] := d[v] + len[v, j];
                  prev[j] := v;
```

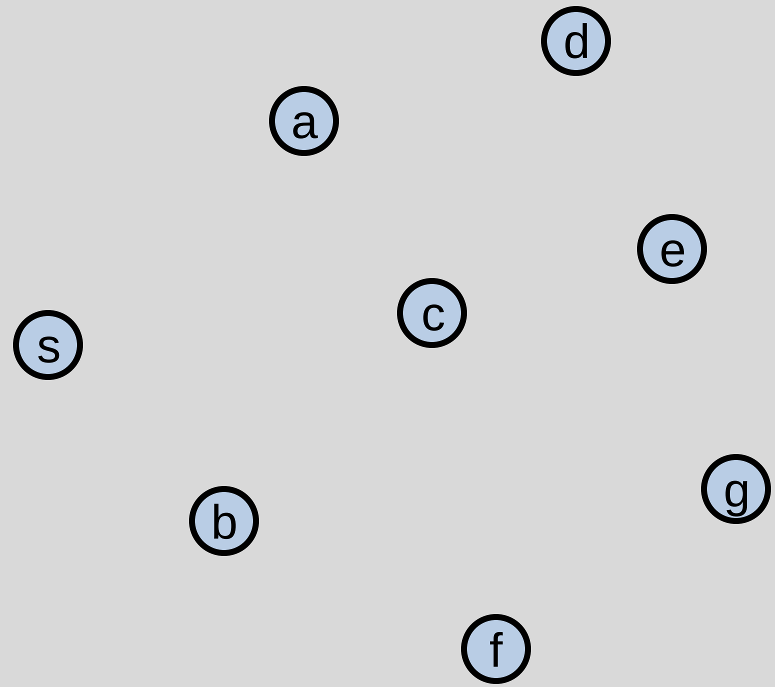
Bottleneck Shortest Path

- Define the bottleneck distance for a path to be the maximum cost edge along the path



Compute the bottleneck shortest paths





How do you adapt Dijkstra's algorithm to handle bottleneck distances

- Does the correctness proof still apply?