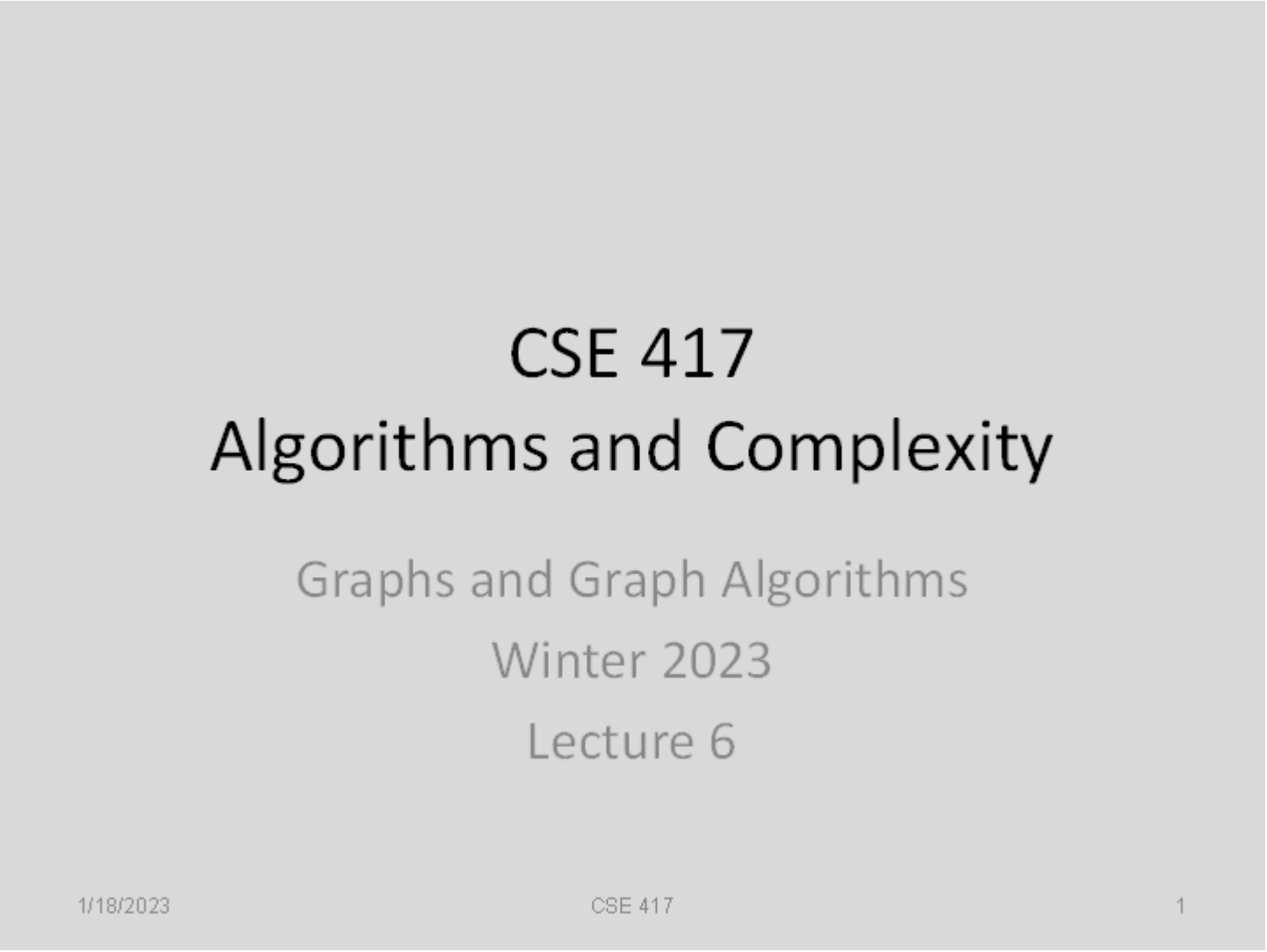
# Lecture06


CSE 417
Algorithms and Complexity
Graphs and Graph Algorithms
Winter 2023
Lecture 6
1/18/2023
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# Announcements

- Reading
  - Chapter 3
  - Start on Chapter 4
- Homework 2
  - Programming problem: related to analysis of stable matching

# Graph Theory

- G = (V, E)
  - V: vertices, |V| = n
  - E: edges, |E| = m
- Undirected graphs
  - Edges sets of two vertices {u, v}
- Directed graphs
  - Edges ordered pairs (u, v)
- Many other flavors
  - Edge / vertices weights
  - Parallel edges
  - Self loops

- Path: $v_1, v_2, \ldots, v_k$, with $(v_i, v_{i+1})$ in E
  - **Simple Path**
  - **Cycle**
  - **Simple Cycle**
- Neighborhood
  - N(v)
- Distance
- Connectivity
  - **Undirected**
  - **Directed (strong connectivity)**
- Trees
  - **Rooted**
  - **Unrooted**

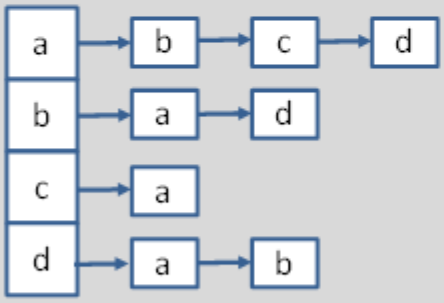
# Graph Representation

Density

$m \leq n^2$


V = { a, b, c, d}

E = { {a, b}, {a, c}, {a, d}, {b, d} }




Adjacency List

O(n + m) space

| | | | |
|---|---|---|---|
| | 1 | 1 | 1 |
| 1 | | 0 | 1 |
| 1 | 0 | | 0 |
| 1 | 1 | 0 | |

Incidence Matrix

O($n^2$) space

# Implementation Issues

- Graph with n vertices, m edges
- Operations
  - Lookup edge
  - Add edge
  - Enumeration edges
  - Initialize graph
- Space requirements

| | A list (sparse) | Matrix (dense) |
|---|---|---|
| Lookup edge | O(n) | O(1) |
| Add edge | O(1) | O(1) |
| Enumeration edges | O(m+n) | O(n²) |
| Initialize graph | O(n) | O(n²) |
| Space requirements | O(n+m) | O(n²) |

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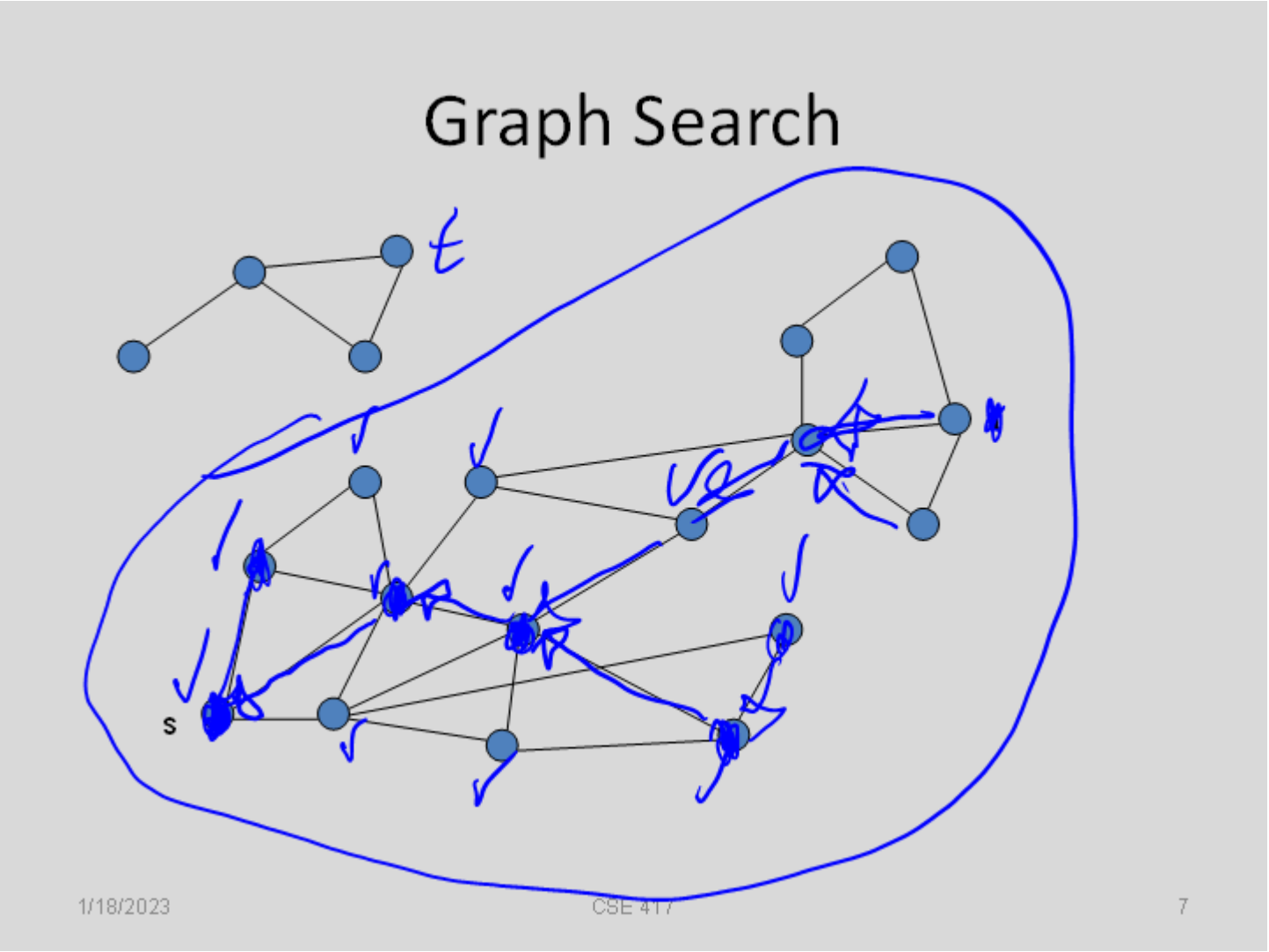
# Graph search

- Find a path from s to t

```
S = {s}
while S is not empty
        u = Select(S)
        visit u
        foreach v in N(u)
                if v is unvisited
                        Add(S, v)
                        Pred[v] = u
                if (v == t) then path found
```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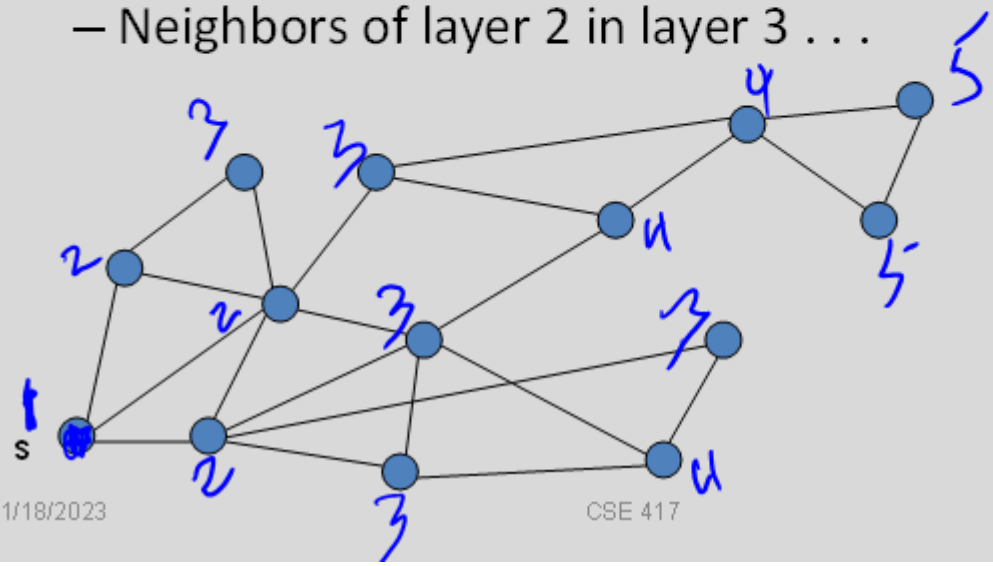
# Breadth first search

- Explore vertices in layers
  - s in layer 1
  - Neighbors of s in layer 2
  - Neighbors of layer 2 in layer 3 . . .



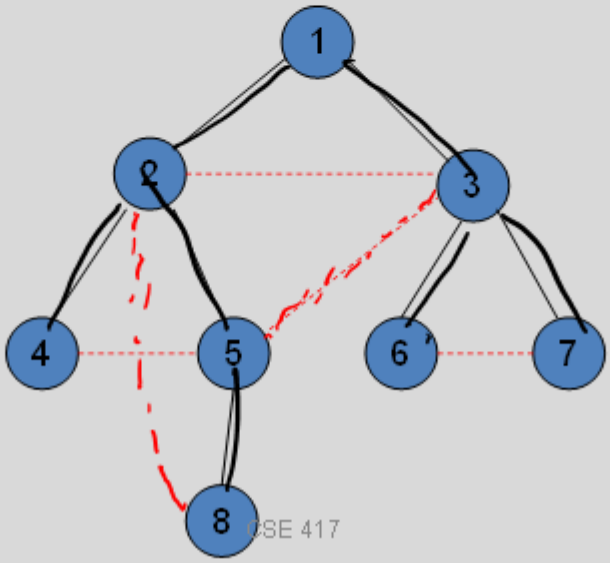
# Breadth First Search

- Build a BFS tree from s

```
Initialize Level[v] = -1 for all v;
Q = {s}
Level[s] = 1;
while Q is not empty
        u = Q.Dequeue()
        foreach v in N(u)
                if (Level[v] == -1)
                        Q.Enqueue(v)
                        Pred[v] = u
                        Level[v] = Level[u] + 1
```

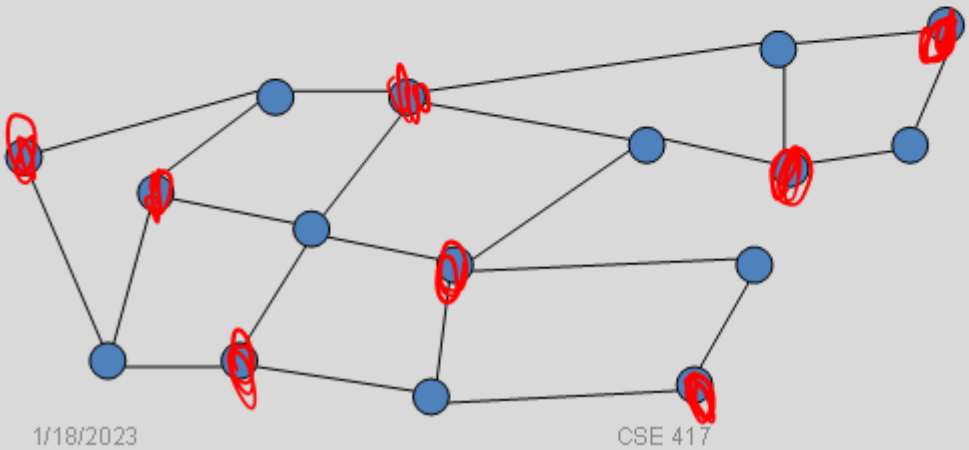
# Key observation

- All edges go between vertices on the same layer or adjacent layers



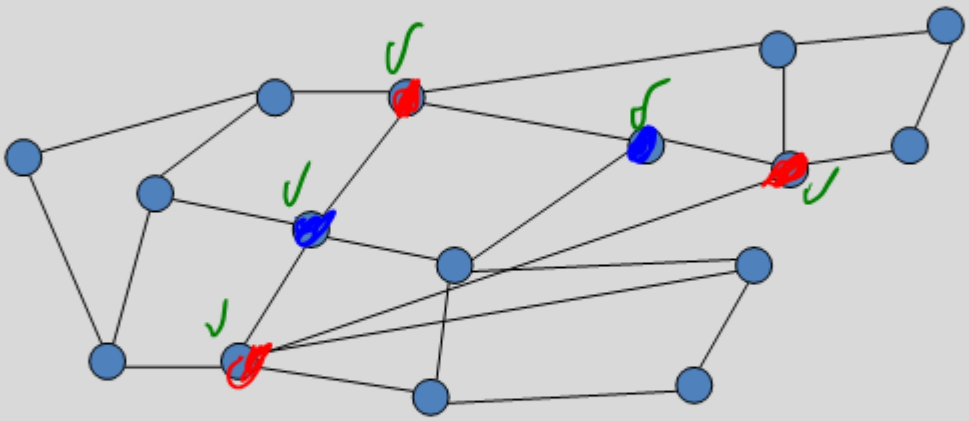
# Bipartite Graphs

- A graph V is bipartite if V can be partitioned into $V_1$, $V_2$ such that all edges go between $V_1$ and $V_2$
- A graph is bipartite if it can be two colored



NO!

# Can this graph be two colored?



# Algorithm

- Run BFS
- Color odd layers red, even layers blue
- If no edges between the same layer, the graph is bipartite
- If edge between two vertices of the same layer, then there is an odd cycle, and the graph is not bipartite

Theorem: A graph is bipartite if and only if it has no odd cycles

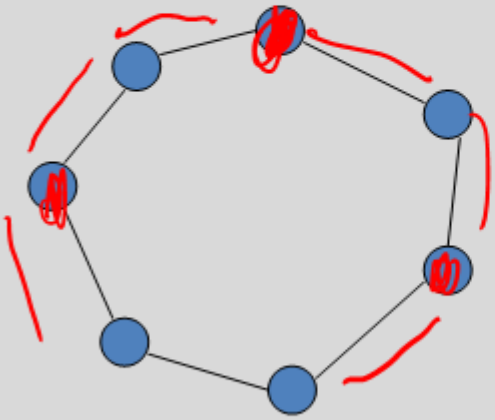
odd cycle $\Rightarrow$ $\neg$ Bipartite

$\neg$ odd cycle $\Rightarrow$ Bipartite

# Lemma 1

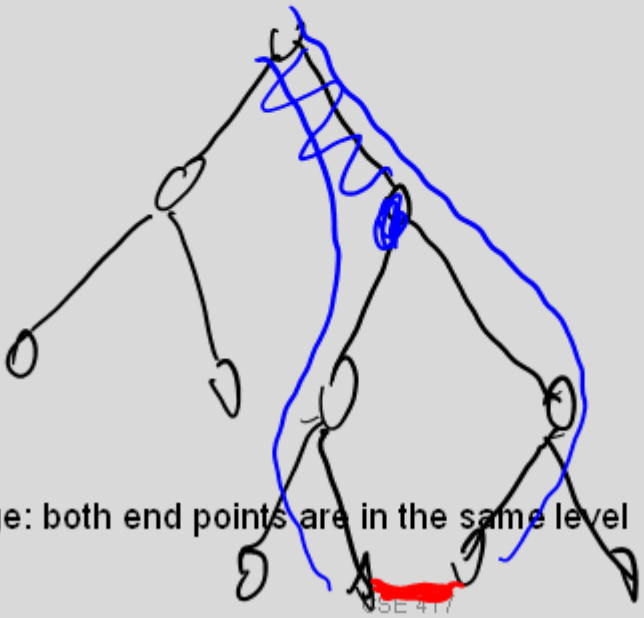
2K+1

- If a graph contains an odd cycle, it is not bipartite



# Lemma 2

- If a BFS tree has an *intra-level edge*, then the graph has an odd length cycle



Intra-level edge: both end points are in the same level

# Lemma 3

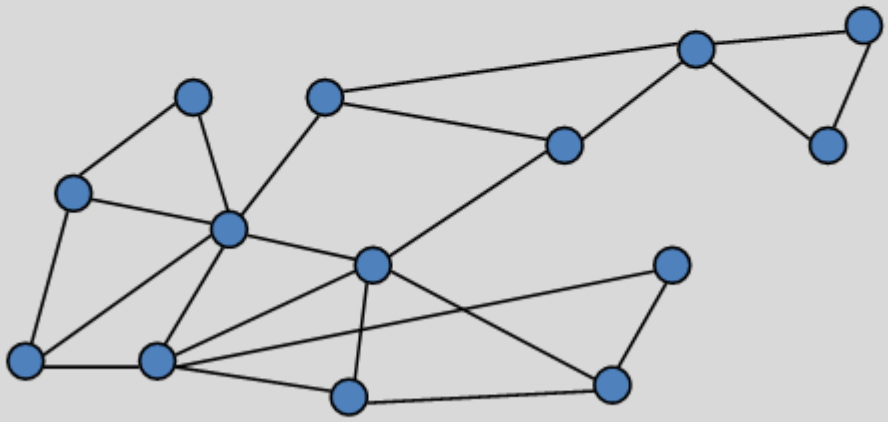
- If a graph has no odd length cycles, then it is bipartite

⇒ BFS has no interlevel edges

Color odd levels red even layers blue.

# Graph Search

- Data structure for next vertex to visit determines search order



# Graph search

Queue

**Breadth First Search**

```
S = {s}
while S is not empty
    u = Dequeue(S)
    if u is unvisited
        visit u
        foreach v in N(u)
            Enqueue(S, v)
```

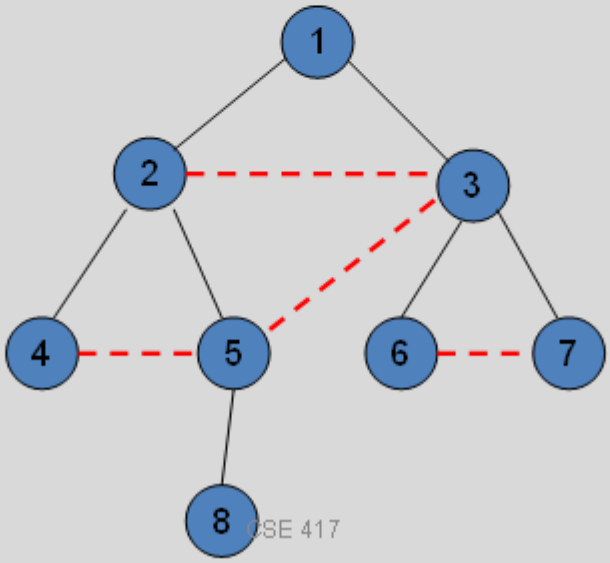
Stack

**Depth First Search**

```
S = {s}
while S is not empty
    u = Pop(S)
    if u is unvisited
        visit u
        foreach v in N(u)
            Push(S, v)
```

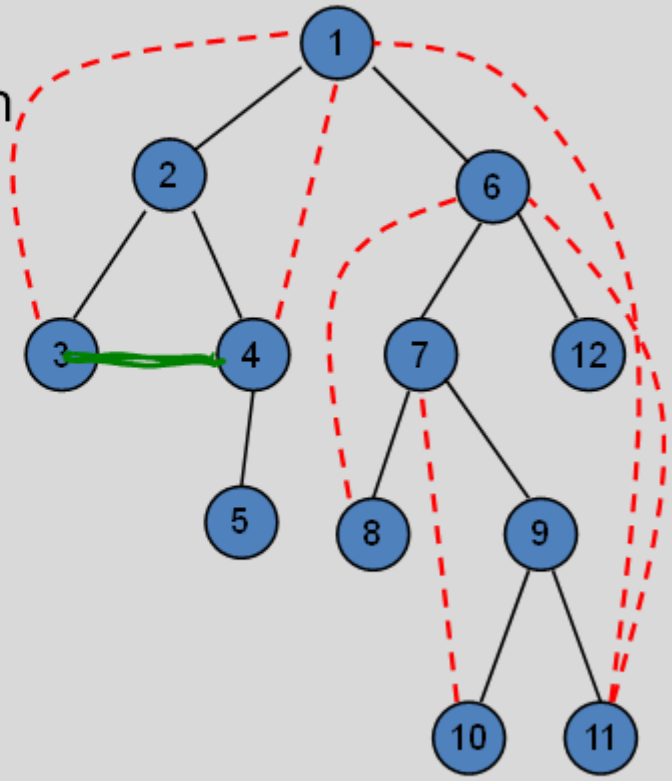
# Breadth First Search

- All edges go between vertices on the same layer or adjacent layers



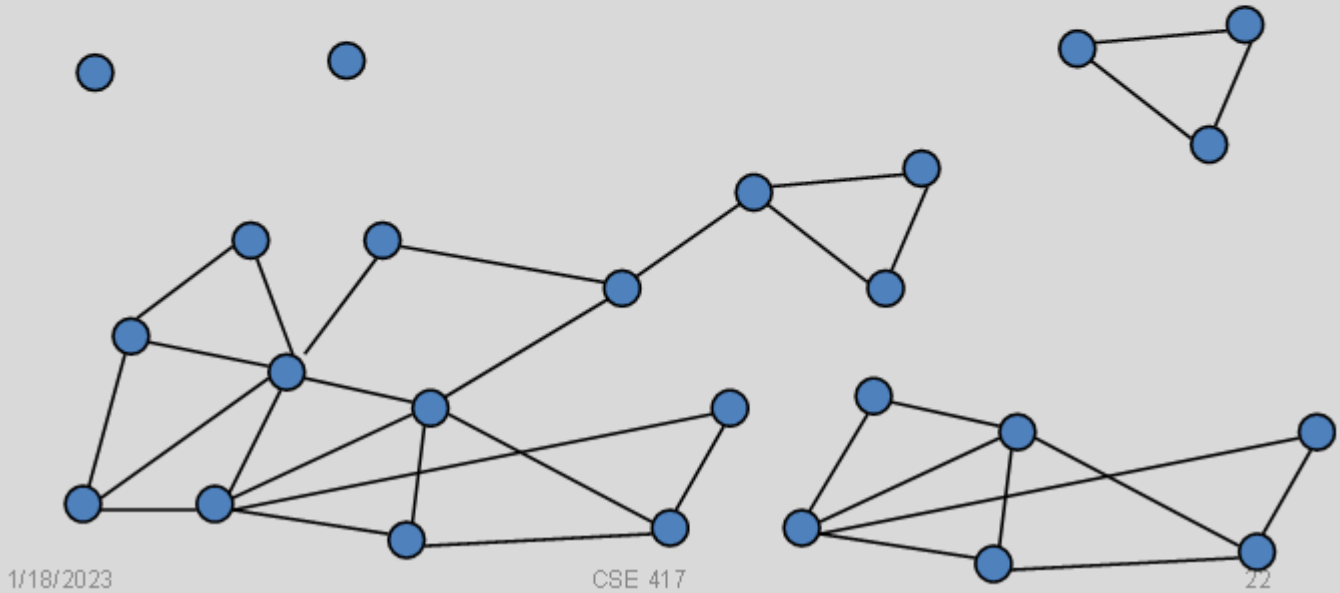
# Depth First Search

- Each edge goes between vertices on the same branch
- No cross edges



# Connected Components

- Undirected Graphs

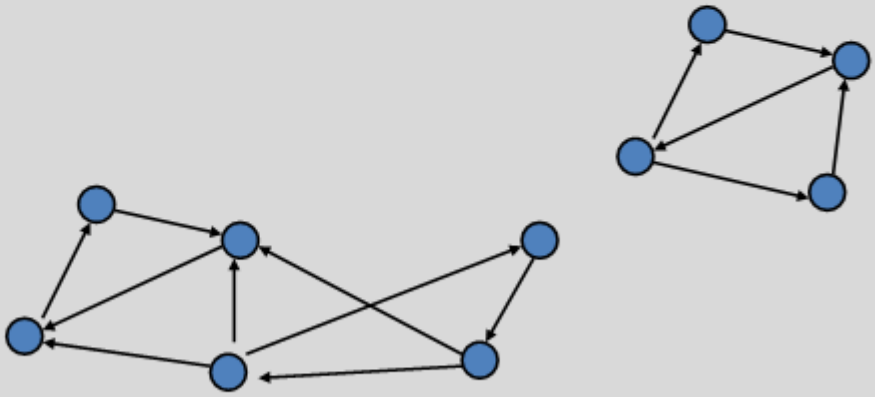


# Computing Connected Components in O(n+m) time

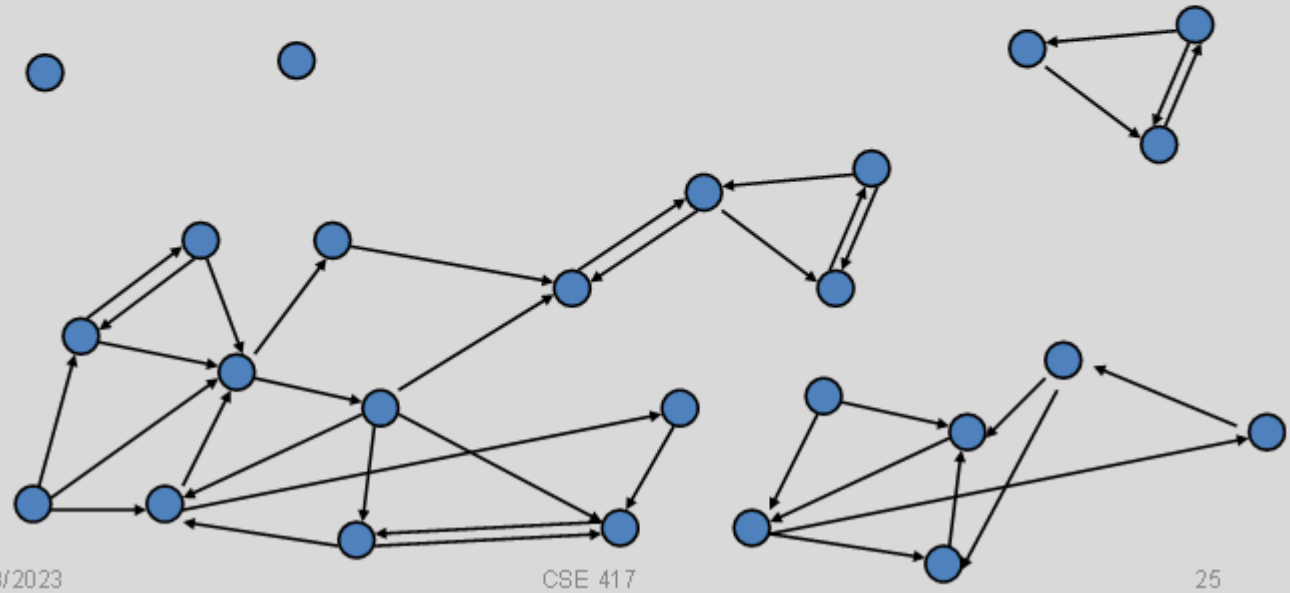
- A search algorithm from a vertex v can find all vertices in v's component
- While there is an unvisited vertex v, search from v to find a new component

# Directed Graphs

- A Strongly Connected Component is a subset of the vertices with paths between every pair of vertices.



# Identify the Strongly Connected Components

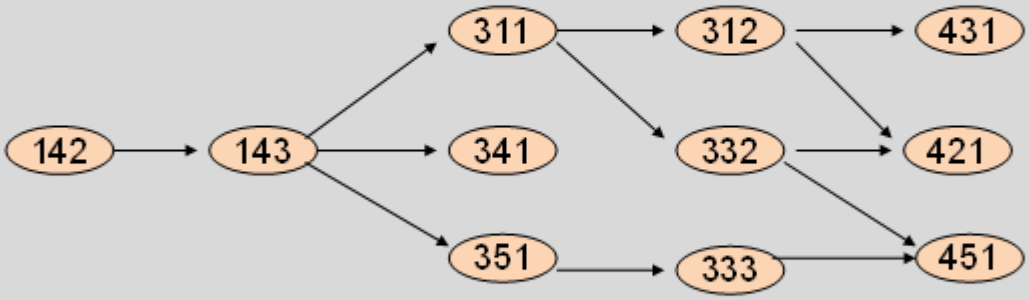


# Strongly connected components can be found in O(n+m) time

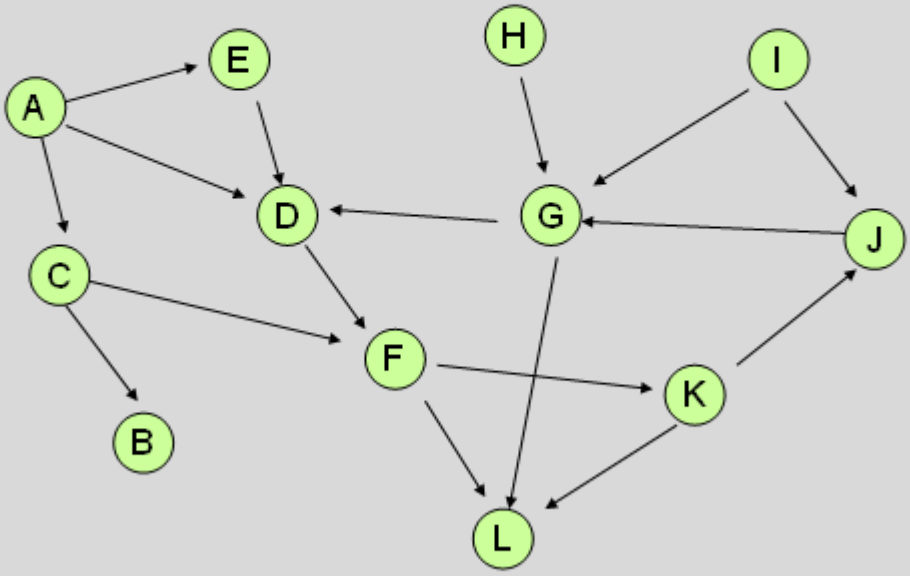
- But it's tricky!
- Simpler problem: given a vertex v, compute the vertices in v's scc in O(n+m) time

# Topological Sort

- Given a set of tasks with precedence constraints, find a linear order of the tasks

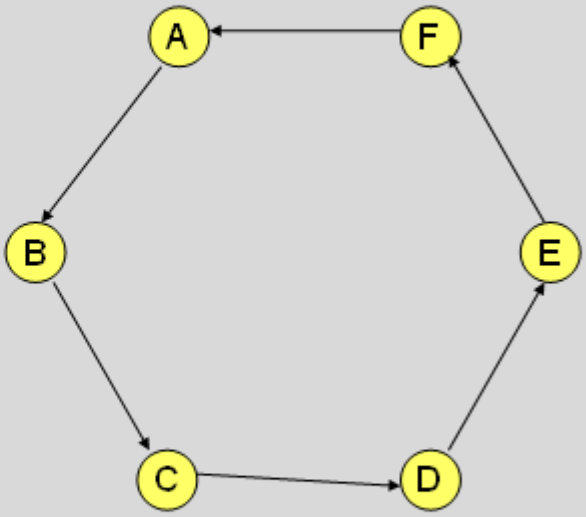


# Find a topological order for the following graph



# If a graph has a cycle, there is no topological sort

- Consider the first vertex on the cycle in the topological sort
- It must have an incoming edge




Definition: A graph is Acyclic if it has no cycles

# Lemma: If a (finite) graph is acyclic, it has a vertex with in-degree 0

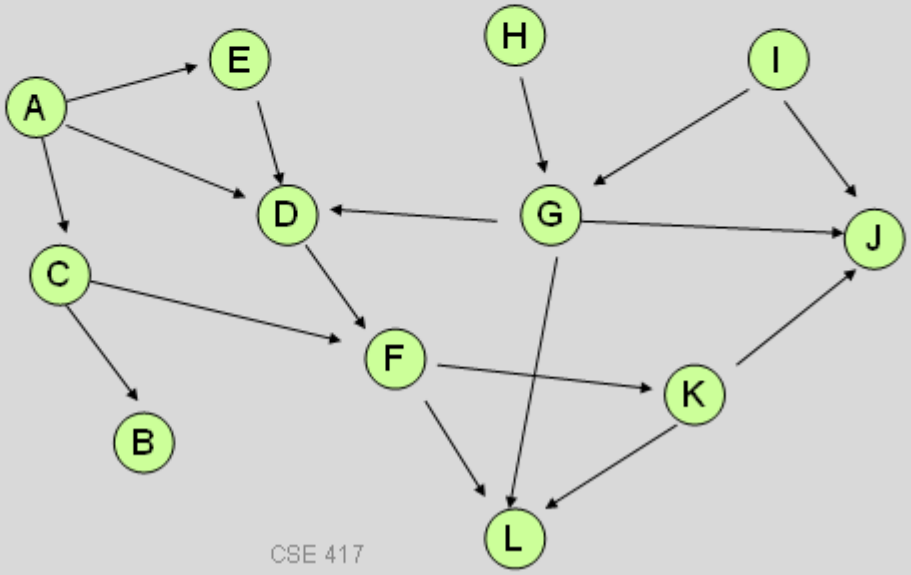
- Proof:
  - Pick a vertex $v_1$, if it has in-degree 0 then done
  - If not, let $(v_2, v_1)$ be an edge, if $v_2$ has in-degree 0 then done
  - If not, let $(v_3, v_2)$ be an edge . . .
  - If this process continues for more than n steps, we have a repeated vertex, so we have a cycle

# Topological Sort Algorithm

While there exists a vertex v with in-degree 0

Output vertex v

Delete the vertex v and all out going edges



# Details for O(n+m) implementation

- Maintain a list of vertices of in-degree 0
- Each vertex keeps track of its in-degree
- Update in-degrees and list when edges are removed
- m edge removals at O(1) cost each