CSE 417
Algorithms and Complexity

Graphs and Graph Algorithms

Winter 2023

Lecture 6

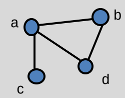
Announcements

- Reading
  - Chapter 3
  - Start on Chapter 4
- Homework 2
  - Programming problem: related to analysis of stable matching

Graph Theory

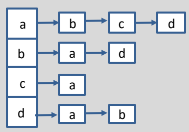
- G = (V, E)
  - V: vertices, |V|= n
  - E: edges, |E| = m
- Undirected graphs
  - Edges sets of two vertices {u, v}
- Directed graphs
  - Edges ordered pairs (u, v)
- Many other flavors
  - Edge / vertices weights
  - Parallel edges
  - Self loops
- Path: $v_1, v_2, \ldots, v_k$, with (v_i, v_{i+1}) in E
  - Simple Path
  - Cycle
  - Simple Cycle
- Neighborhood
  - N(v)
- Distance
- Connectivity
  - Undirected
  - Directed (strong connectivity)
- Trees
  - Rooted
  - Unrooted

Graph Representation



V = { a, b, c, d}

E = { {a, b}, {a, c}, {a, d}, {b, d} }



Adjacency List

O(n + m) space

	1	1	1
1		0	1
1	0		0
1	1	0	

Incidence Matrix

$O(n^2)$ space

Implementation Issues

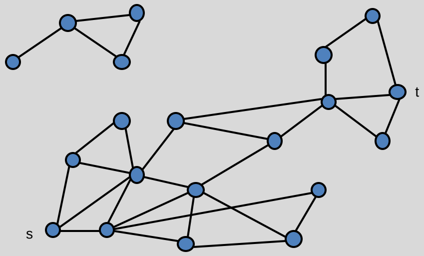
- Graph with n vertices, m edges
- Operations
  - Lookup edge
  - Add edge
  - Enumeration edges
  - Initialize graph
- Space requirements

Graph search

- Find a path from s to t

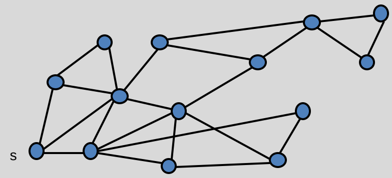
```
S = {s}
while S is not empty
        u = Select(S)
        visit u
        foreach v in N(u)
                if v is unvisited
                        Add(S, v)
                        Pred[v] = u
                if (v == t) then path found
```

Graph Search



Breadth first search

- Explore vertices in layers
  - s in layer 1
  - Neighbors of s in layer 2
  - Neighbors of layer 2 in layer 3 . . .



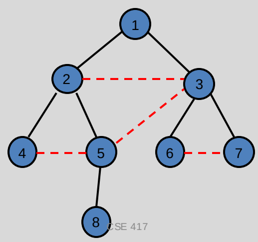
Breadth First Search

- Build a BFS tree from s

```
Initialize Level[v] = -1 for all v;
Q = {s}
Level[s] = 1;
while Q is not empty
        u = Q.Dequeue()
        foreach v in N(u)
                if (Level[v] == -1)
                        Q.Enqueue(v)
                        Pred[v] = u
                        Level[v] = Level[u] + 1
```

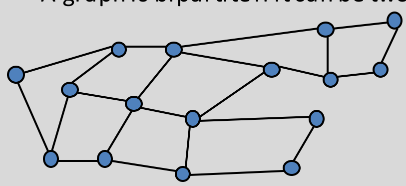

Key observation

- All edges go between vertices on the same layer or adjacent layers

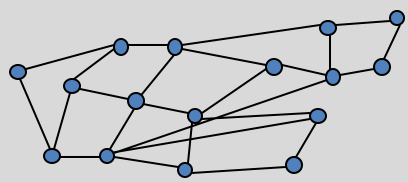


Bipartite Graphs

- A graph V is bipartite if V can be partitioned into V_1, V_2 such that all edges go between V_1 and V_2
- A graph is bipartite if it can be two colored



Can this graph be two colored?



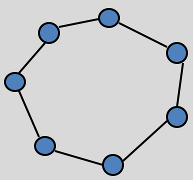
Algorithm

- Run BFS
- Color odd layers red, even layers blue
- If no edges between the same layer, the graph is bipartite
- If edge between two vertices of the same layer, then there is an odd cycle, and the graph is not bipartite

Theorem: A graph is bipartite if and only if it has no odd cycles

Lemma 1

- If a graph contains an odd cycle, it is not bipartite



Lemma 2

- If a BFS tree has an *intra-level edge*, then the graph has an odd length cycle

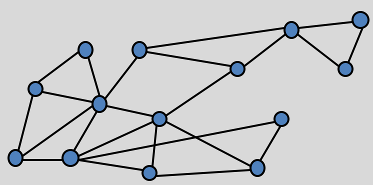
Intra-level edge: both end points are in the same level

Lemma 3

- If a graph has no odd length cycles, then it is bipartite

Graph Search

- Data structure for next vertex to visit determines search order



Graph search

Breadth First Search

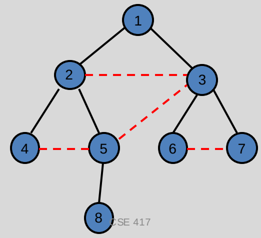
```
S = {s}
while S is not empty
    u = Dequeue(S)
    if u is unvisited
        visit u
        foreach v in N(u)
            Enqueue(S, v)
```

Depth First Search

```
S = {s}
while S is not empty
    u = Pop(S)
    if u is unvisited
        visit u
        foreach v in N(u)
            Push(S, v)
```

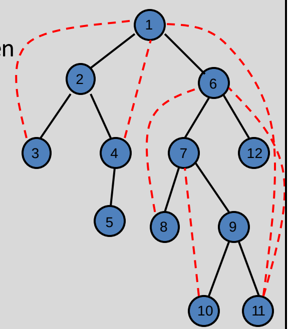

Breadth First Search

- All edges go between vertices on the same layer or adjacent layers



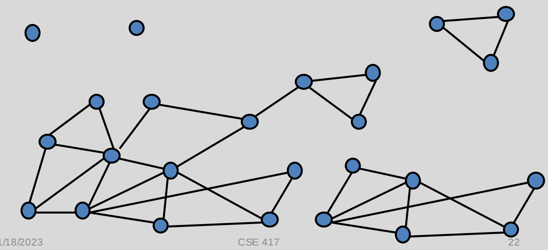
Depth First Search

- Each edge goes between vertices on the same branch
- No cross edges



Connected Components

- Undirected Graphs

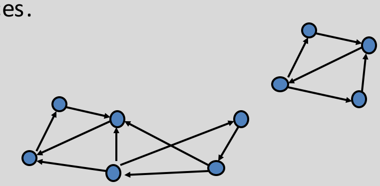


Computing Connected Components in O(n+m) time

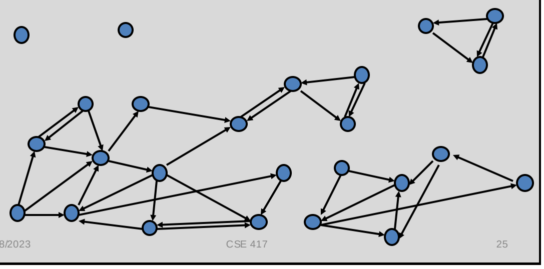
- A search algorithm from a vertex v can find all vertices in v's component
- While there is an unvisited vertex v, search from v to find a new component

Directed Graphs

- A Strongly Connected Component is a subset of the vertices with paths between every pair of vertices.



Identify the Strongly Connected Components

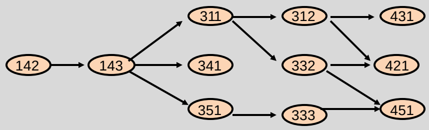


Strongly connected components can be found in O(n+m) time

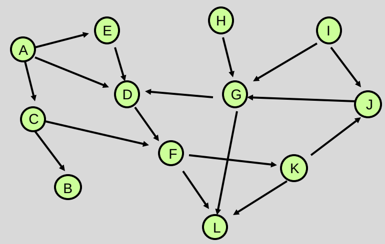
- But it's tricky!
- Simpler problem: given a vertex v, compute the vertices in v's scc in O(n+m) time

Topological Sort

- Given a set of tasks with precedence constraints, find a linear order of the tasks

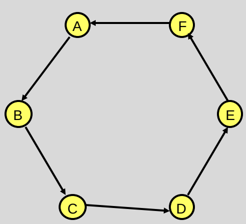


Find a topological order for the following graph



If a graph has a cycle, there is no topological sort

- Consider the first vertex on the cycle in the topological sort
- It must have an incoming edge



Definition: A graph is Acyclic if it has no cycles

Lemma: If a (finite) graph is acyclic, it has a vertex with in-degree 0

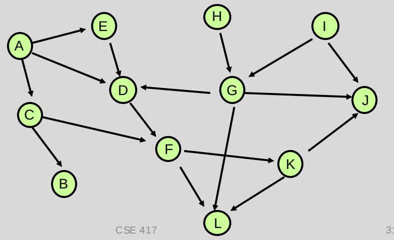
- Proof:
  - Pick a vertex v_1, if it has in-degree 0 then done
  - If not, let (v_2, v_1) be an edge, if v_2 has in-degree 0 then done
  - If not, let (v_3, v_2) be an edge . . .
  - If this process continues for more than n steps, we have a repeated vertex, so we have a cycle

Topological Sort Algorithm

While there exists a vertex v with in-degree 0

Output vertex v

Delete the vertex v and all out going edges



Details for O(n+m) implementation

- Maintain a list of vertices of in-degree 0
- Each vertex keeps track of its in-degree
- Update in-degrees and list when edges are removed
- m edge removals at O(1) cost each