CSE 417
Algorithms and Computational Complexity

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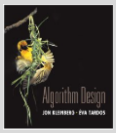
Winter 2023

Lecture 2

Announcements

- Course website
  - https://courses.cs.washington.edu/courses/cse417/23wi/
- Homework due Fridays
  - HW 1, Due Friday, January 13, 11:59 pm
  - Submit solutions via gradescope
- Class discussion through edstem discussion board

Course Mechanics



- Homework
  - Due Fridays
  - About 5 problems, sometimes programming
  - Programming – your choice of language
  - Target: 1 week turnaround on grading
- Exams – In class
  - MT – Wednesday, February 8
  - Final – Monday, March 13
- Approximate grade weighting
  - HW: 50, MT: 15, Final: 35
- Course web
  - Slides, Handouts
- Instructor Office hours (CSE2 344)
  - Tuesday, 2:30-3:30 pm, Thursday, 4:00-5:00 pm

TA Office Hours

- Nickolay Perezhogin,
  Office hours: Thursday, 10:00 AM - 12:00 PM (noon) (CSE1 5th Floor Breakout)
- Artin Tajdini,
  Office hours: Wednesday, 2:30 PM - 4:30 PM (CSE1 220)
- Tom Zhaoyang Tian,
  Office hours: Wednesday, 12:30 PM - 1:30 PM (CSE2 150); Friday, 4:30 PM - 5:30 PM (CSE2 150)
- Michael Wen,
  Office hours: Monday, 11:30 AM - 12:30 PM (CSE2 153); Thursday, 1:30 PM - 2:30 PM (CSE2 153)
- Albert Weng,
  Office hours: Tuesday, 10:30 AM - 11:30 AM (CSE2 153); Friday, 9:30 AM - 10:30 AM (CSE2 153)
- Yilin Zhang,
  Office hours: Monday, 3:30 PM - 5:30 PM (CSE2 150)

Stable Matching: Formal Problem

- Input
  - Preference lists for $m_1, m_2, \ldots, m_n$
  - Preference lists for $w_1, w_2, \ldots, w_n$
- Output
  - Perfect matching M satisfying stability property (e.g., no instabilities) :

For all m', m'', w', w''
    If (m', w') $\in$ M and (m'', w'') $\in$ M then
        (m' prefers w' to w'') or (w'' prefers m'' to m')

Idea for an Algorithm

m proposes to w
    If w is unmatched, w accepts
    If w is matched to m_2
        If w prefers m to m_2, w accepts m, dumping m_2
        If w prefers m_2 to m, w rejects m

Unmatched m proposes to the highest w on its preference list that it has not already proposed to

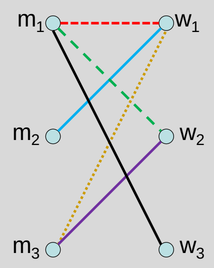
Algorithm

Initially all m in M and w in W are free
While there is a free m
  w highest on m's list that m has not proposed to
  if w is free, then match (m, w)
  else
    suppose (m_2, w) is matched
    if w prefers m to m_2
      unmatch (m_2, w)
      match (m, w)

Example

m_1: w_1 w_2 w_3
m_2: w_1 w_3 w_2
m_3: w_1 w_2 w_3

w_1: m_2 m_3 m_1
w_2: m_3 m_1 m_2
w_3: m_3 m_1 m_2



Order: m_1, m_2, m_3, m_1, m_3, m_1

Does this work?

- Does it terminate?
- Is the result a stable matching?

- Begin by identifying invariants and measures of progress
  - m's proposals get worse (have higher m-rank)
  - Once w is matched, w stays matched
  - w's partners get better (have lower w-rank)

Claim: If an m reaches the end of its list, then all the w's are matched

Claim: The algorithm stops in at most n^2 steps

When the algorithms halts, every w is matched

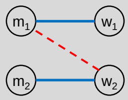
Hence, the algorithm finds a perfect matching

The resulting matching is stable

Suppose

$(m_1, w_1) \in M$, $(m_2, w_2) \in M$

m_1 prefers w_2 to w_1



How could this happen?

Result

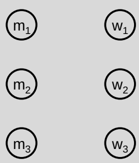
- Simple, $O(n^2)$ algorithm to compute a stable matching
- Corollary
  - A stable matching always exists

A closer look

Stable matchings are not necessarily fair

m_1: w_1 w_2 w_3
m_2: w_2 w_3 w_1
m_3: w_3 w_1 w_2

w_1: m_2 m_3 m_1
w_2: m_3 m_1 m_2
w_3: m_1 m_2 m_3



How many stable matchings can you find?

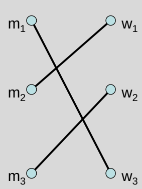
Algorithm under specified

- Many different ways of picking m's to propose
- Surprising result
  - All orderings of picking free m's give the same result
- Proving this type of result
  - Reordering argument
  - Prove algorithm is computing something mores specific
    - Show property of the solution – so it computes a specific stable matching

M-rank and W-rank of matching

- m-rank: position of matching w in preference list
- M-rank: sum of m-ranks
- w-rank: position of matching m in preference list
- W-rank: sum of w-ranks

m_1: w_1 w_2 w_3
m_2: w_1 w_3 w_2
m_3: w_1 w_2 w_3
w_1: m_2 m_3 m_1
w_2: m_3 m_1 m_2
w_3: m_3 m_1 m_2



What is the M-rank?

What is the W-rank?

Suppose there are n m's, and n w's

- What is the minimum possible M-rank?
- What is the maximum possible M-rank?
- Suppose each m is matched with a random w, what is the expected M-rank?

Random Preferences

Suppose that the preferences are completely random

m_1: w_8 w_3 w_1 w_5 w_9 w_2 w_4 w_6 w_7 w_{10}
m_2: w_7 w_{10} w_1 w_9 w_3 w_4 w_8 w_2 w_5 w_6
…
w_1: m_1 m_4 m_9 m_5 m_{10} m_3 m_2 m_6 m_8 m_7
w_2: m_5 m_8 m_1 m_3 m_2 m_7 m_9 m_{10} m_4 m_6
…

If there are n m's and n w's, what is the expected value of the M-rank and the W-rank when the proposal algorithm computes a stable matching?

Stable Matching Algorithms

- M Proposal Algorithm
  - Iterate over all m's until all are matched
- W Proposal Algorithm
  - Change the role of m's and w's
  - Iterate over all w's until all are matched

Generating a random permutation

```
public static int[] Permutation(int n, Random rand) {
    int[] arr = IdentityPermutation(n);

    for (int i = 1; i < n; i++) {
        int j = rand.Next(0, i + 1);
        int temp = arr[i];
        arr[i] = arr[j];
        arr[j] = temp;
    }
    return arr;
}
```

What is the run time of the Stable Matching Algorithm?

Initially all m in M and w in W are free
While there is a free m **Executed at most n^2 times**
    w highest on m's list that m has not proposed to
    if w is free, then match (m, w)
    else
        suppose (m_2, w) is matched
        if w prefers m to m_2
            unmatch (m_2, w)
            match (m, w)

O(1) time per iteration

- Find free m
- Find next available w
- If w is matched, determine m_2
- Test if w prefer m to m_2
- Update matching

What does it mean for an algorithm to be efficient?

Key ideas

- Formalizing real world problem
  - Model: graph and preference lists
  - Mechanism: stability condition
- Specification of algorithm with a natural operation
  - Proposal
- Establishing termination of process through invariants and progress measure
- Under specification of algorithm
- Establishing uniqueness of solution