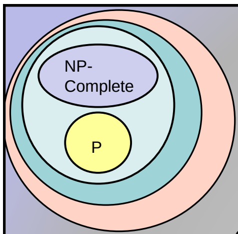
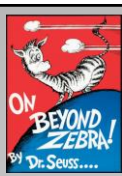



# CSE 417
# Algorithms and Complexity

Autumn 2023

Lecture 29

NP-Completeness and Beyond

## Announcements

- Final Exam: Monday, December 11, 8:30 AM
  - One Hour Fifty Minutes
  - Comprehensive (but roughly 2/3rds post midterm)
  - Topics will include: recurrences, dynamic programming, graph algorithms, network flow

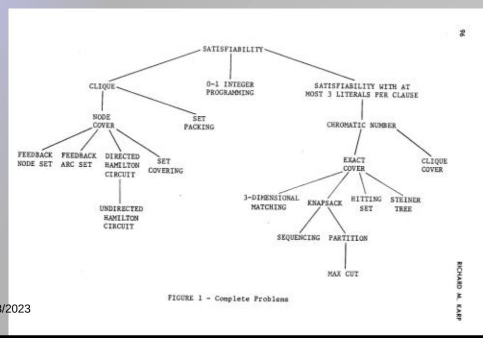
| | |
|---|---|
| ~~Fri, Dec 1~~ | ~~Net Flow Applications~~ |
| ~~Mon, Dec 4~~ | ~~Net Flow Applications + NP-Completeness~~ |
| ~~Wed, Dec 6~~ | ~~NP-Completeness~~ |
| Fri, Dec 8 | NP-Completeness and Beyond |
| Mon, Dec 11 | Final Exam |

## NP-Completeness Proofs

- Prove that problem X is NP-Complete
  - Show that X is in NP (usually easy)
  - Pick a known NP complete problem Y
  - Show Y $<_P$ X

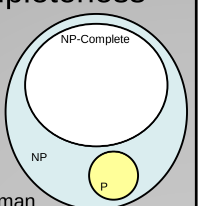


## Reducibility Among Combinatorial Problems






## Populating the NP-Completeness Universe




- Circuit Sat $<_P$ 3-SAT
- 3-SAT $<_P$ Independent Set
- 3-SAT $<_P$ Vertex Cover
- Independent Set $<_P$ Clique
- 3-SAT $<_P$ Hamiltonian Circuit
- Hamiltonian Circuit $<_P$ Traveling Salesman
- 3-SAT $<_P$ Integer Linear Programming
- 3-SAT $<_P$ Graph Coloring
- 3-SAT $<_P$ Subset Sum
- Subset Sum $<_P$ Scheduling with Release times and deadlines



## Coping with NP-Completeness

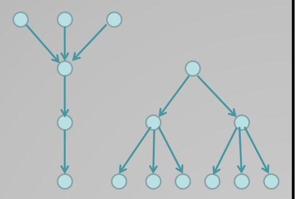
- Approximation Algorithms
- Exact solution via Branch and Bound
- Local Search

I can't find an efficient algorithm, but neither can all these famous people.

# Multiprocessor Scheduling

- Unit execution tasks
- Precedence graph
- K-Processors

- Polynomial time for k=2
- Open for k = constant
- NP-complete is k is part of the problem





# Highest level first is 2-Optimal

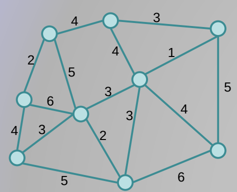
Choose k items on the highest level

Claim: number of rounds is at least twice the optimal.



# Christofides TSP Algorithm

- Undirected graph satisfying triangle inequality

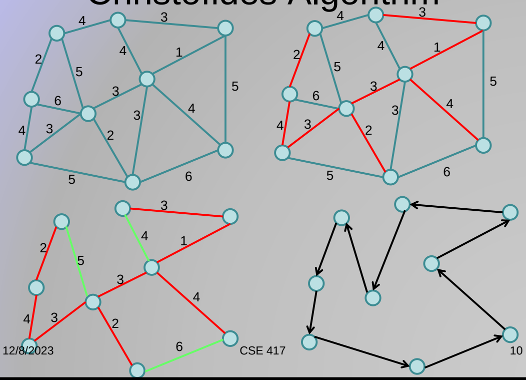


1. Find MST
2. Add additional edges so that all vertices have even degree
3. Build Eulerian Tour

3/2 Approximation



# Christofides Algorithm





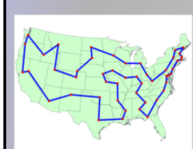
# Branch and Bound

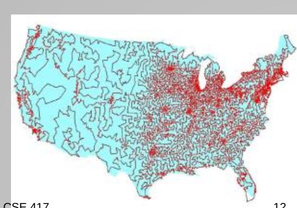
- Brute force search – tree of all possible solutions
- Branch and bound – compute a lower bound on all possible extensions
  - Prune sub-trees that cannot be better than optimal



# Branch and Bound for TSP

- Enumerate all possible paths
- Lower bound, Current path cost plus MST of remaining points
- Euclidean TSP
  - Points on the plane with Euclidean Distance
  - Sample data set: State Capitals





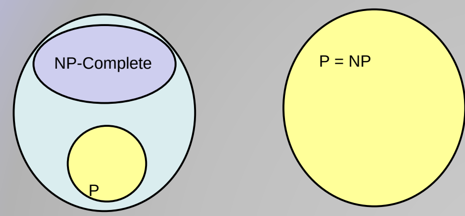
# Local Optimization

- Improve an optimization problem by local improvement
  – Neighborhood structure on solutions
  – Travelling Salesman 2-Opt (or K-Opt)
  – Independent Set Local Replacement



# What we don't know

- P vs. NP

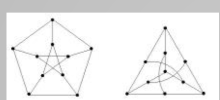





# If P ≠ NP, is there anything in between

- Yes, Ladner [1975]
- Problems not known to be in P or NP Complete
  – Factorization
  – Discrete Log
  – Graph Isomorphism

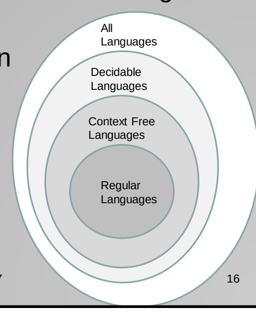
Solve $g^k = b$ over a finite group





# Complexity Theory

- Computational requirements to recognize languages
- Models of Computation
- Resources
- Hierarchies






# Time complexity

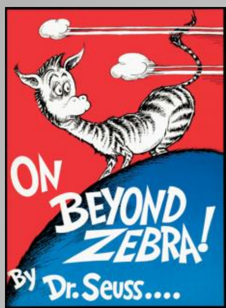
- P: (Deterministic) Polynomial Time
- NP: Non-deterministic Polynomial Time
- EXP: Exponential Time



# Space Complexity

- Amount of Space (Exclusive of Input)
- L: Logspace, problems that can be solved in O(log n) space for input of size n
  – Related to Parallel Complexity
- PSPACE, problems that can be required in a polynomial amount of space

## So what is beyond NP?






## NP vs. Co-NP

- Given a Boolean formula, is it true for some choice of inputs
- Given a Boolean formula, is it true for all choices of inputs



## Problems beyond NP

- Exact TSP, Given a graph with edge lengths and an integer K, does the minimum tour have length K
- Minimum circuit, Given a circuit C, is it true that there is no smaller circuit that computes the same function a C



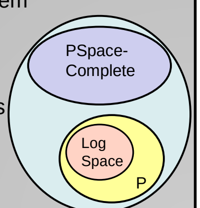
## Polynomial Hierarchy

- Level 1
  - $\exists X_1\ \Phi(X_1)$, $\forall X_1\ \Phi(X_1)$
- Level 2
  - $\forall X_1 \exists X_2\ \Phi(X_1,X_2)$, $\exists X_1 \forall X_2\ \Phi(X_1,X_2)$
- Level 3
  - $\forall X_1 \exists X_2 \forall X_3\ \Phi(X_1,X_2,X_3)$, $\exists X_1 \forall X_2 \exists X_3\ \Phi(X_1,X_2,X_3)$



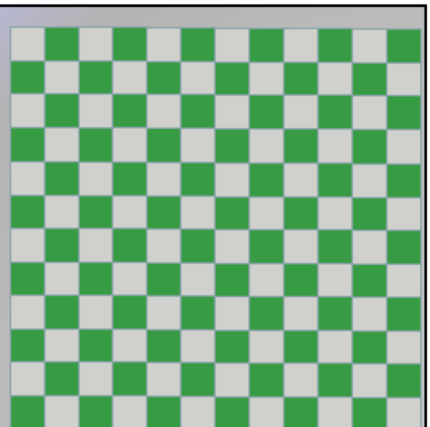
## Polynomial Space

- Quantified Boolean Expressions
  - $\exists X_1 \forall X_2 \exists X_3 \ldots \exists X_{n-1} \forall X_n\ \Phi(X_1,X_2,X_3 \ldots X_{n-1}X_n)$
- Space bounded games
  - Competitive Facility Location Problem
  - N x N Chess
- Counting problems
  - The number of Hamiltonian Circuits






## N X N Chess



# Even Harder Problems

```
public int[] RecolorSwap(int[] coloring) {
        int k = maxColor(coloring);

        for (int v = 0; v < nVertices; v++) {
            if (coloring[v] == k) {
                int[] nbdColorCount = ColorCount(v, k, coloring);
                List<Edge> edges1 = vertices[v].Edges;

                foreach (Edge e1 in edges1) {
                    int w = e1.Head;
                    if (nbdColorCount[coloring[w]] == 1)
                        if (RecolorSwap(v, w, k, coloring))
                            break;
                }
            }
        }
        return coloring;
    }
```

Is this code correct?



# Halting Problem

- Given a program P that does not take any inputs, does P eventually exit?



# Impossibility of solving the Halting Problem

Suppose Halt(P) returns true if P halts, and false otherwise

Consider the program G:

```
Define G {
    if (Halt(G)){
        while (true) ;
    }
    else {
        exit();
    }
}
```

What is Halt(G)?



# Undecidable Problems

- The Halting Problem is undecidable
- Impossible problems are hard . . .