

CSE 417
Algorithms and Complexity

Autumn 2023

Lecture 27

Network Flow Applications

NP-Completeness

Announcements

- Homework 9
- Exam practice problems on course homepage
- Final Exam: Monday, December 11, 8:30 AM
  – One Hour Fifty Minutes

Fri, Dec 1	Net Flow Applications
Mon, Dec 4	Net Flow Applications + NP-Completeness
Wed, Dec 6	NP-Completeness
Fri, Dec 8	NP-Completeness
Mon, Dec 11	Final Exam

Problem Reduction

- Reduce Problem A to Problem B
  – Convert an instance of Problem A to an instance of Problem B
  – Use a solution of Problem B to get a solution to Problem A
- Practical
  – Use a program for Problem B to solve Problem A
- Theoretical
  – Show that Problem B is at least as hard as Problem A

Minimum Cut Applications

- Image Segmentation
- Open Pit Mining / Task Selection Problem
- Reduction to Min Cut problem

S, T is a cut if S, T is a partition of the vertices with s in S and t in T

The capacity of an S, T cut is the sum of the capacities of all edges going from S to T

Image Segmentation

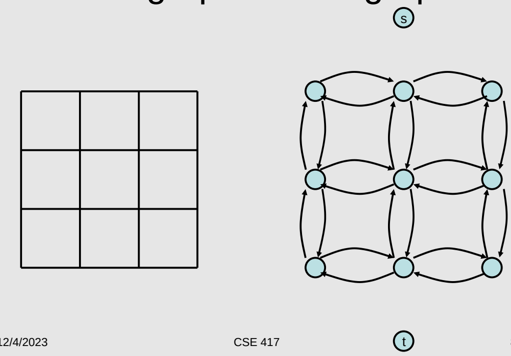
- Separate foreground from background



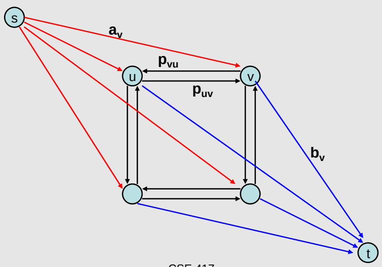
Image analysis

- a_i: value of assigning pixel i to the foreground
- b_i: value of assigning pixel i to the background
- p_{ij}: penalty for assigning i to the foreground, j to the background or vice versa
- A: foreground, B: background
- $Q(A,B) = \Sigma_{\{i \text{ in } A\}} a_i + \Sigma_{\{j \text{ in } B\}} b_j - \Sigma_{\{(i,j) \text{ in } E, \text{ i in } A, \text{ j in } B\}} p_{ij}$

Pixel graph to flow graph



Mincut Construction

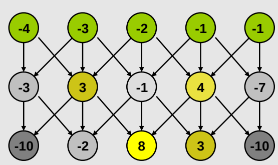


Open Pit Mining
(Task selection)

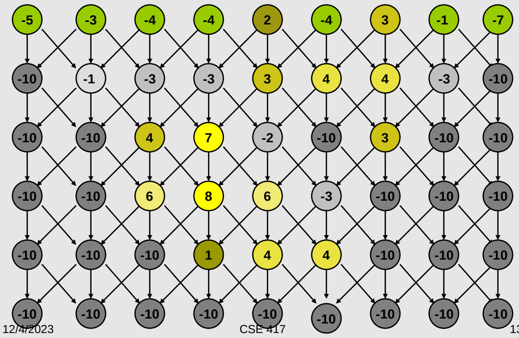
Open Pit Mining

- Each unit of earth has a profit (possibly negative)
- Getting to the ore below the surface requires removing the dirt above
- Test drilling gives reasonable estimates of costs
- Plan an optimal mining operation

Mine Graph

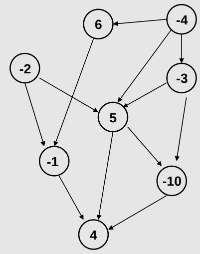


Determine an optimal mine



Generalization

- Precedence graph G=(V,E)
- Each v in V has a profit p(v)
- A set F is *feasible* if when w in F, and (v,w) in E, then v in F.
- Find a feasible set to maximize the profit

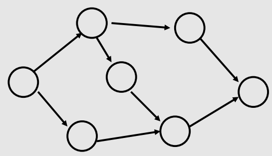


Min cut algorithm for profit maximization

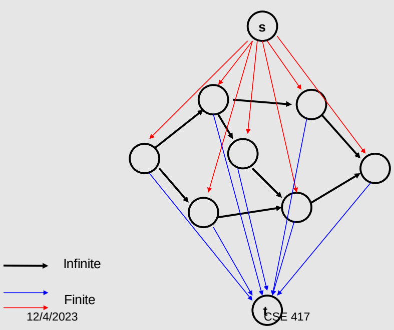
- Construct a flow graph where the minimum cut identifies a feasible set that maximizes profit

Precedence graph construction

- Precedence graph G=(V,E)
- Each edge in E has infinite capacity
- Add vertices s, t
- Each vertex in V is attached to s and t with finite capacity edges

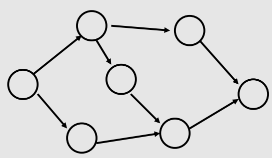


Find a finite value cut with at least two vertices on each side of the cut



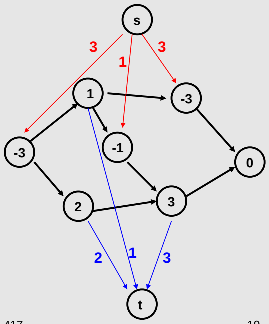
The sink side of a finite cut is a feasible set

- No edges permitted from S to T
- If a vertex is in T, all of its ancestors are in T

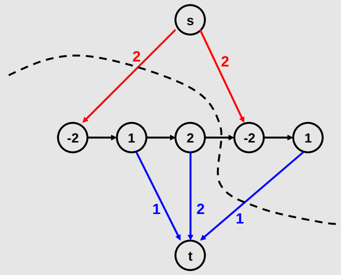


Setting the costs

- If p(v) > 0,
  - cap(v,t) = p(v)
  - cap(s,v) = 0
- If p(v) < 0
  - cap(s,v) = -p(v)
  - cap(v,t) = 0
- If p(v) = 0
  - cap(s,v) = 0
  - cap(v,t) = 0



Minimum cut gives optimal solution Why?

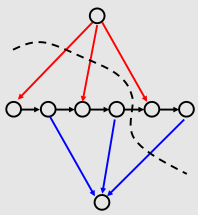


Computing the Profit

- $\text{Cost}(W) = \Sigma_{\{w \text{ in } W;\ p(w) < 0\}} -p(w)$
- $\text{Benefit}(W) = \Sigma_{\{w \text{ in } W;\ p(w) > 0\}}\ p(w)$
- Profit(W) = Benefit(W) – Cost(W)

- Maximum cost and benefit
  - C = Cost(V)
  - B = Benefit(V)

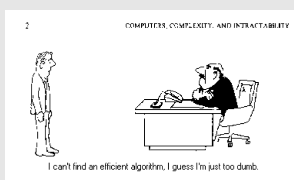
Express Cap(S,T) in terms of B, C, Cost(T), Benefit(T), and Profit(T)



Cap(S,T) = Cost(T) + Ben(S) = Cost(T) + Ben(S) + Ben(T) – Ben(T)
= B + Cost(T) – Ben(T) = B – Profit(T)


NP-Completeness


NP Completeness

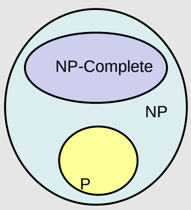


Algorithms vs. Lower bounds

- Algorithmic Theory
  - What we can compute
    - I can solve problem X with resources R
  - Proofs are almost always to give an algorithm that meets the resource bounds
- Lower bounds
  - How do we show that something can't be done?

Theory of NP Completeness

The Universe



Polynomial Time

- P: Class of problems that can be solved in polynomial time
  - Corresponds with problems that can be solved efficiently in practice
  - Right class to work with "theoretically"

Decision Problems

- Theory developed in terms of yes/no problems
  - Independent set
    - Given a graph G and an integer K, does G have an independent set of size at least K
  - Shortest Path
    - Given a graph G with edge lengths, a start vertex s, and end vertex t, and an integer K, does the graph have a path between s and t of length at most K

What is NP?

- Problems solvable in non-deterministic polynomial time . . .
- Problems where "yes" instances have polynomial time checkable certificates

Certificate examples

- Independent set of size K
  - The Independent Set
- Satisfiable formula
  - Truth assignment to the variables
- Hamiltonian Circuit Problem
  - A cycle including all of the vertices
- K-coloring a graph
  - Assignment of colors to the vertices

Certifiers and Certificates: 3-Satisfiability

SAT: Does a given CNF formula have a satisfying formula

Certificate: An assignment of truth values to the n boolean variables

Certifier: Check that each clause has at least one true literal,

instance s

$$(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (\overline{x_1} \vee \overline{x_3} \vee \overline{x_4})$$

certificate t

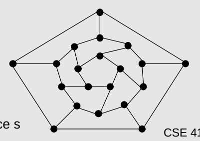
$$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$$

Certifiers and Certificates: Hamiltonian Cycle

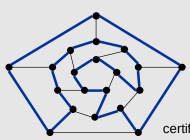
HAM-CYCLE. Given an undirected graph G = (V, E), does there exist a simple cycle C that visits every node?

Certificate. A permutation of the n nodes.

Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.



instance s



certificate t

Polynomial time reductions

- Y is Polynomial Time Reducible to X
  - Solve problem Y with a polynomial number of computation steps and a polynomial number of calls to a black box that solves X
  - Notations: $Y <_P X$
- Usually, this is converting an input of Y to an input for X, solving X, and then converting the answer back

Composability Lemma

- If $X <_P Y$ and $Y <_P Z$ then $X <_P Z$

Lemmas

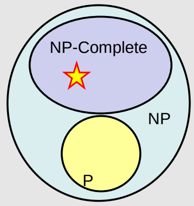
- Suppose $Y <_P X$. If X can be solved in polynomial time, then Y can be solved in polynomial time.
- Suppose $Y <_P X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

NP-Completeness

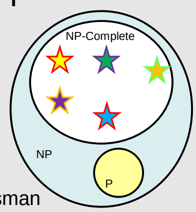
- A problem X is NP-complete if
  - X is in NP
  - For every Y in NP, $Y <_P X$
- X is a “hardest” problem in NP
- If X is NP-Complete, Z is in NP and $X <_P Z$
  - Then Z is NP-Complete

Cook’s Theorem

- There is an NP Complete problem
  - The Circuit Satisfiability Problem



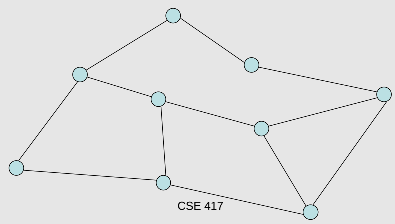
Populating the NP-Completeness Universe



- Circuit Sat $<_P$ 3-SAT
- 3-SAT $<_P$ Independent Set
- 3-SAT $<_P$ Vertex Cover
- Independent Set $<_P$ Clique
- 3-SAT $<_P$ Hamiltonian Circuit
- Hamiltonian Circuit $<_P$ Traveling Salesman
- 3-SAT $<_P$ Integer Linear Programming
- 3-SAT $<_P$ Graph Coloring
- 3-SAT $<_P$ Subset Sum
- Subset Sum $<_P$ Scheduling with Release times and deadlines

Graph Coloring

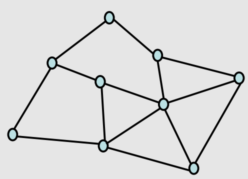
- NP-Complete
  - Graph 3-coloring
- Polynomial
  - Graph 2-Coloring

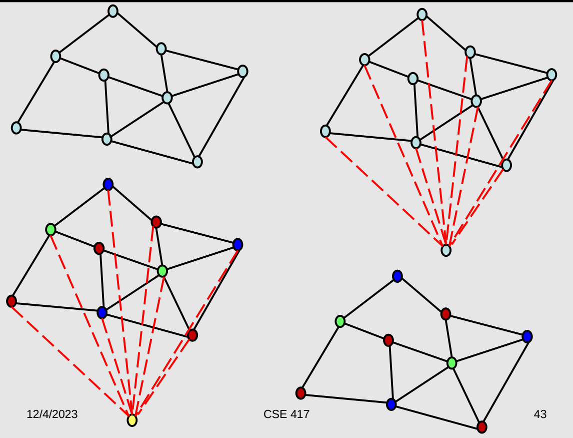


Graph 4-Coloring

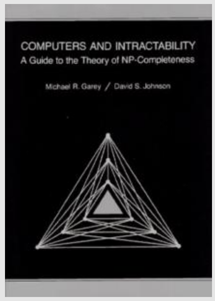
- Given a graph G, can G be colored with 4 colors?
- Prove 4-Coloring is NP Complete
- Proof: 3-Coloring $<_P$ 4-Coloring
- Show that you can 3-Color a graph if you have an algorithm to 4-Color a graph

3-Coloring $<_P$ 4-Coloring





Garey and Johnson



P vs. NP Question