

CSE 417
Algorithms and Complexity

Lecture 25

Autumn 2023

Network Flow, Part 2

Outline

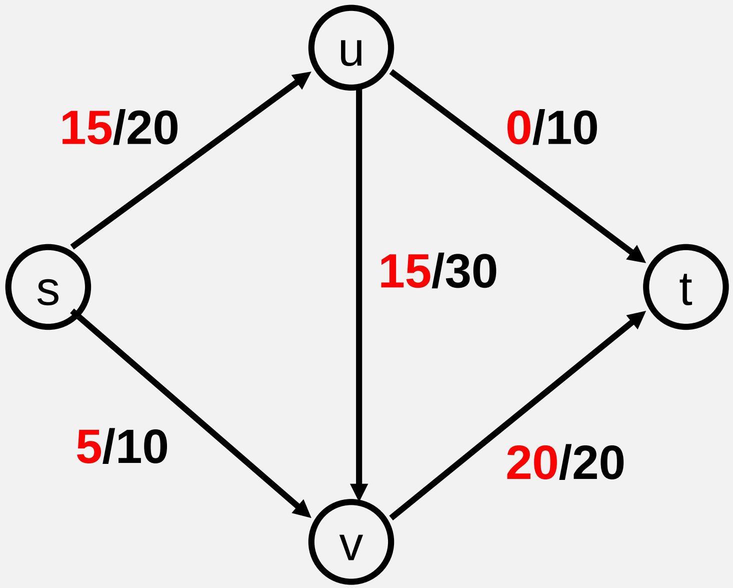
- ~~Network flow definitions~~
- ~~Flow examples~~
- ~~Augmenting Paths~~
- ~~Residual Graph~~
- ~~Ford Fulkerson Algorithm~~
- Cuts
- Maxflow-MinCut Theorem
- Simple applications of Max Flow

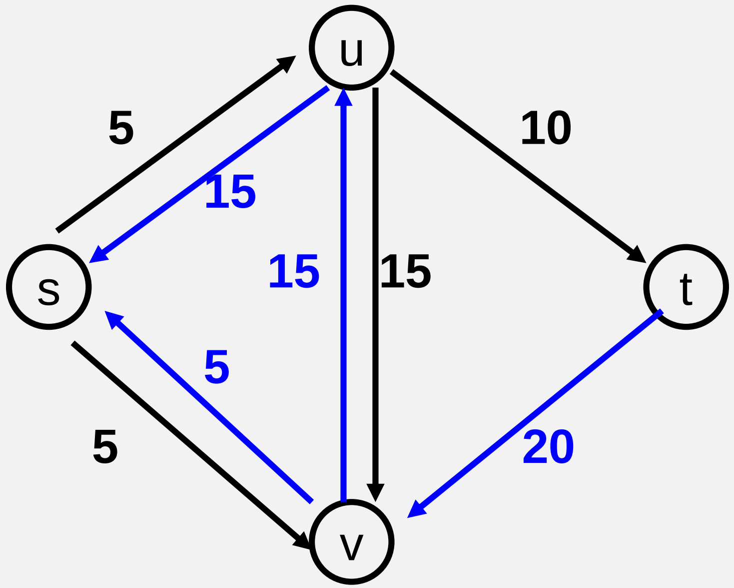
Network Flow Definitions

- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, c(e) >= 0
- Problem, assign flows f(e) to the edges such that:
  - 0 <= f(e) <= c(e)
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is a large as possible

Residual Graph

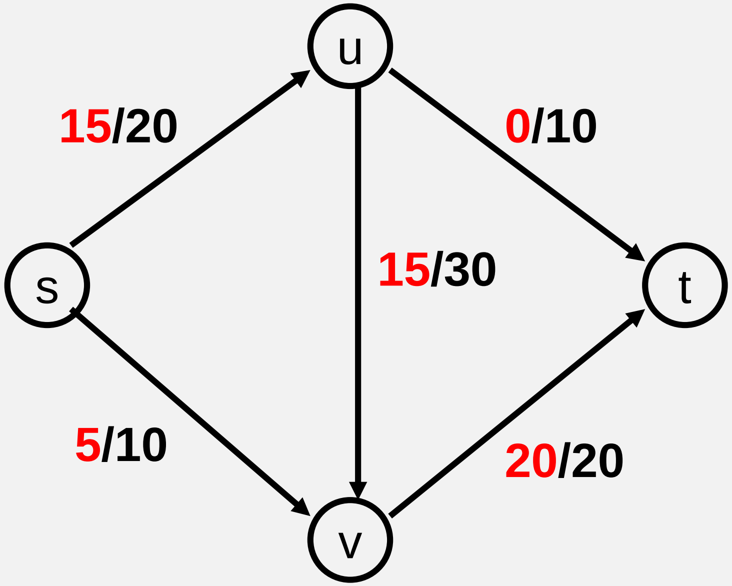
- Flow graph showing the remaining capacity
- Flow graph G, Residual Graph G_R
  - G: edge e from u to v with capacity c and flow f
  - G_R: edge e' from u to v with capacity c – f
  - G_R: edge e'' from v to u with capacity f

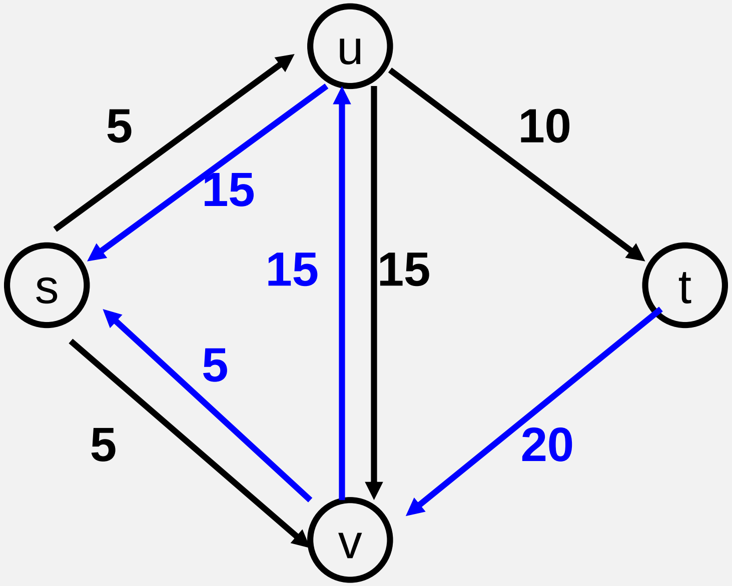




Augmenting Path Algorithm

- Augmenting path in residual graph
  - Vertices $v_1, v_2, \ldots, v_k$
    - $v_1 = s$, $v_k = t$
    - Possible to add b units of flow between v_j and v_{j+1} for j = 1 … k-1





Adding flow along a path in the residual graph

- Let P be an s-t path in the residual graph with capacity b
- b units of flow can be added along P in the graph G
- Need to show:
  - new flow satisfies capacity constraints
  - new flow satisfies conservation constraints

Ford-Fulkerson Algorithm (1956)

while not done

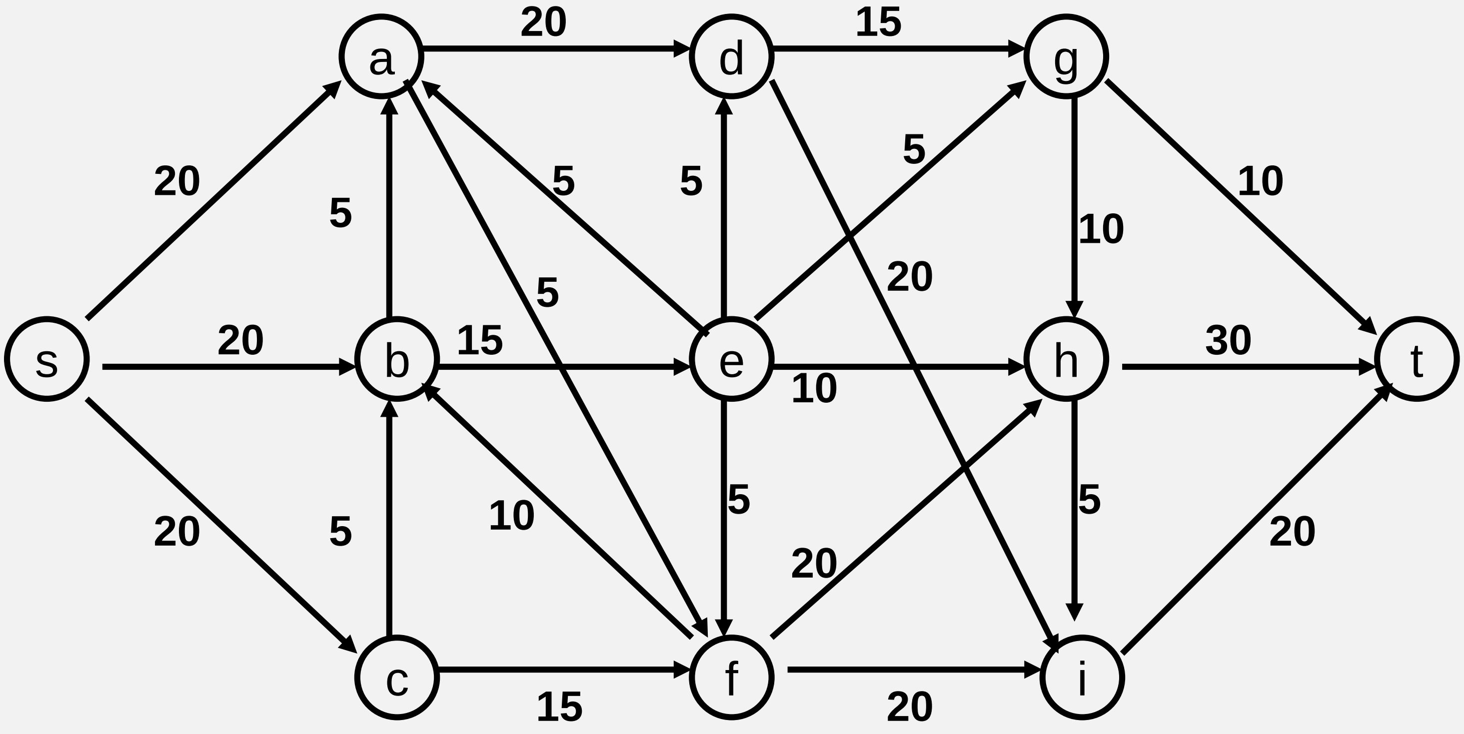
Construct residual graph G_R

Find an s-t path P in G_R with capacity $b > 0$

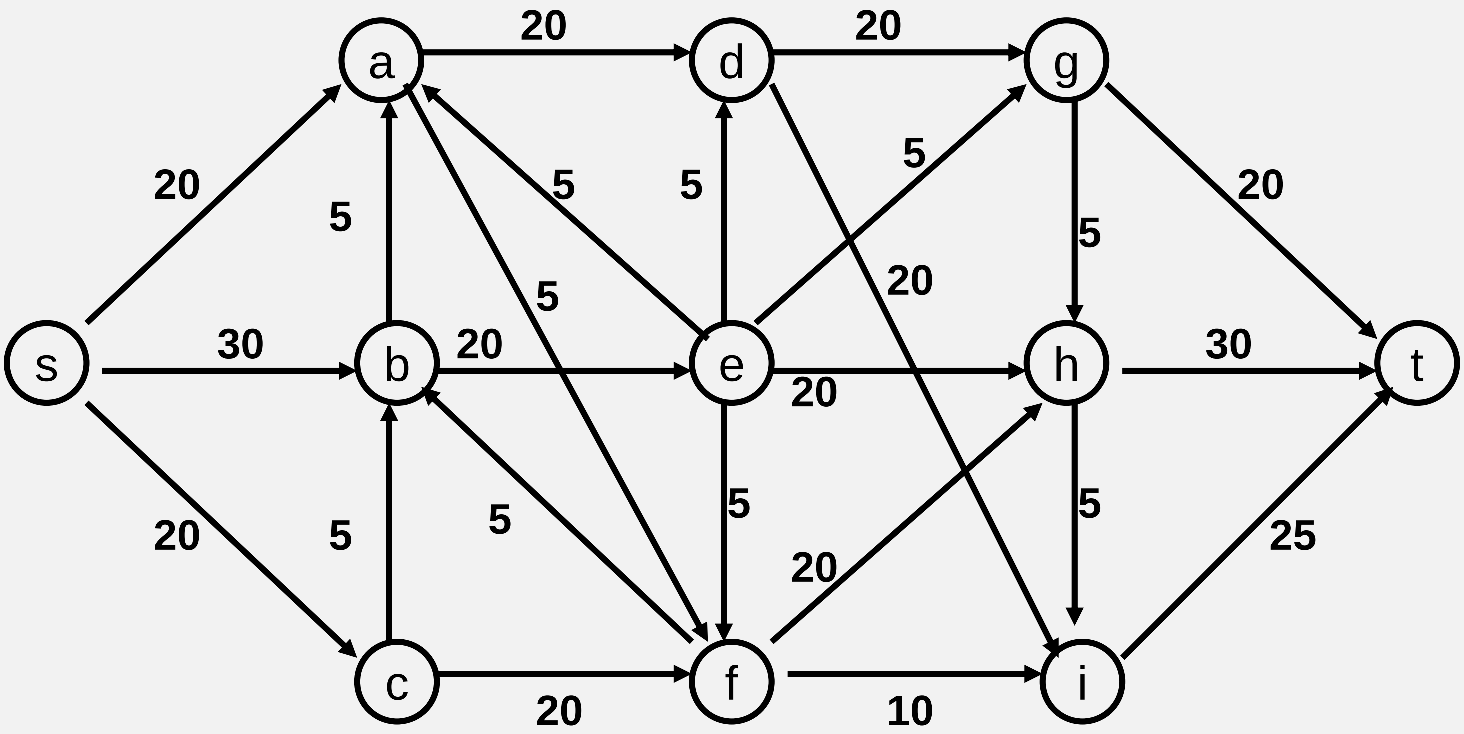
Add b units of flow along path P in G

If the sum of the capacities of edges leaving S is at most C, then the algorithm takes at most C iterations

Flow Example I



Flow Example II

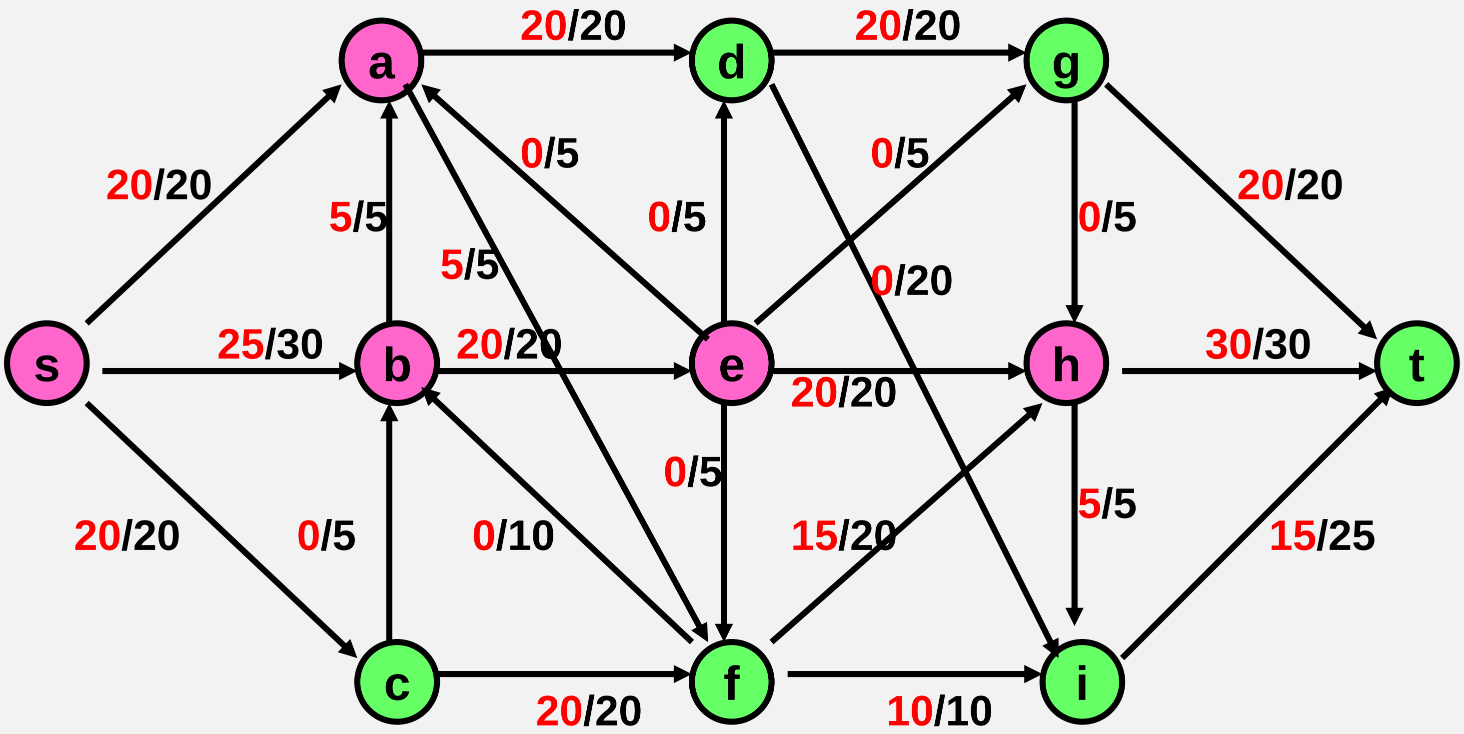


Cuts in a graph

- Cut: Partition of V into disjoint sets S, T with s in S and t in T.
- Cap(S,T): sum of the capacities of edges from S to T
- Flow(S,T): net flow out of S
  - Sum of flows out of S minus sum of flows into S
- Flow(S,T) <= Cap(S,T)

What is Cap(S,T) and Flow(S,T)

S={s, a, b, e, h}, T = {c, f, i, d, g, t}



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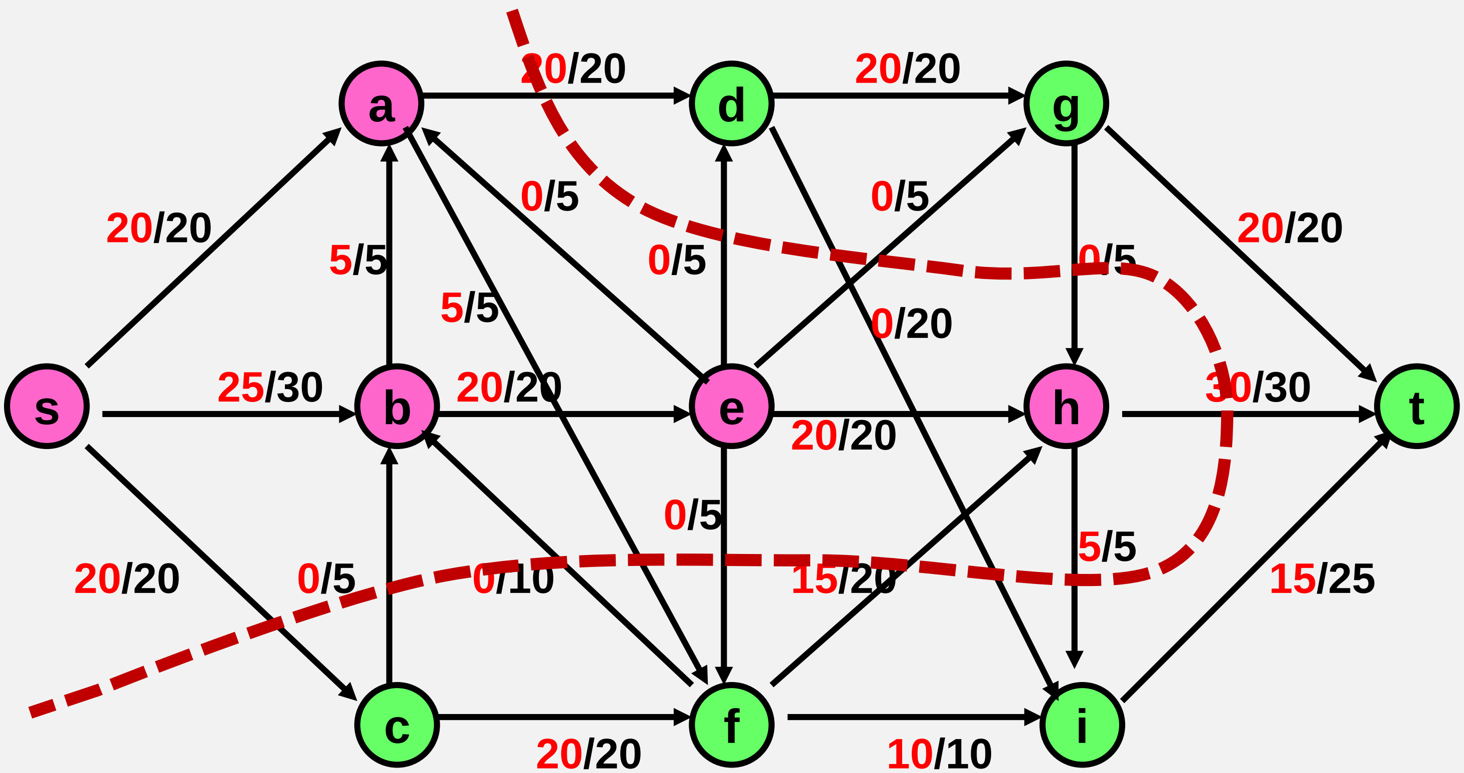
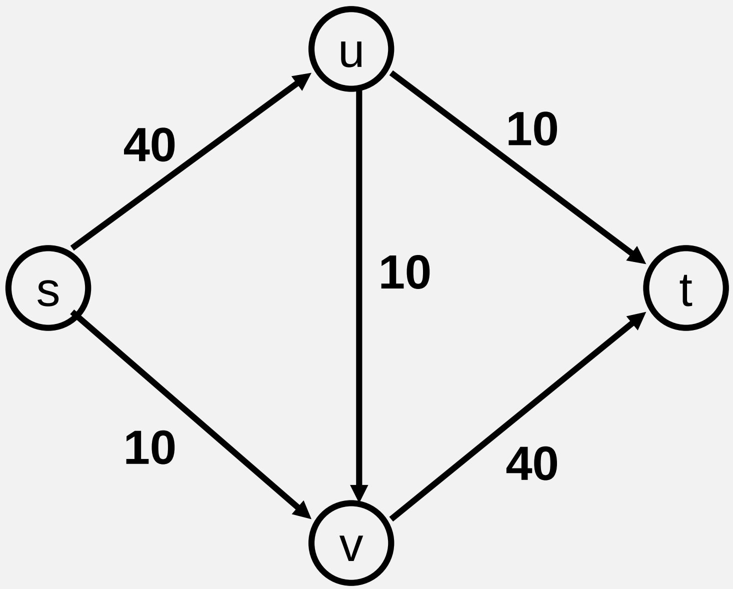
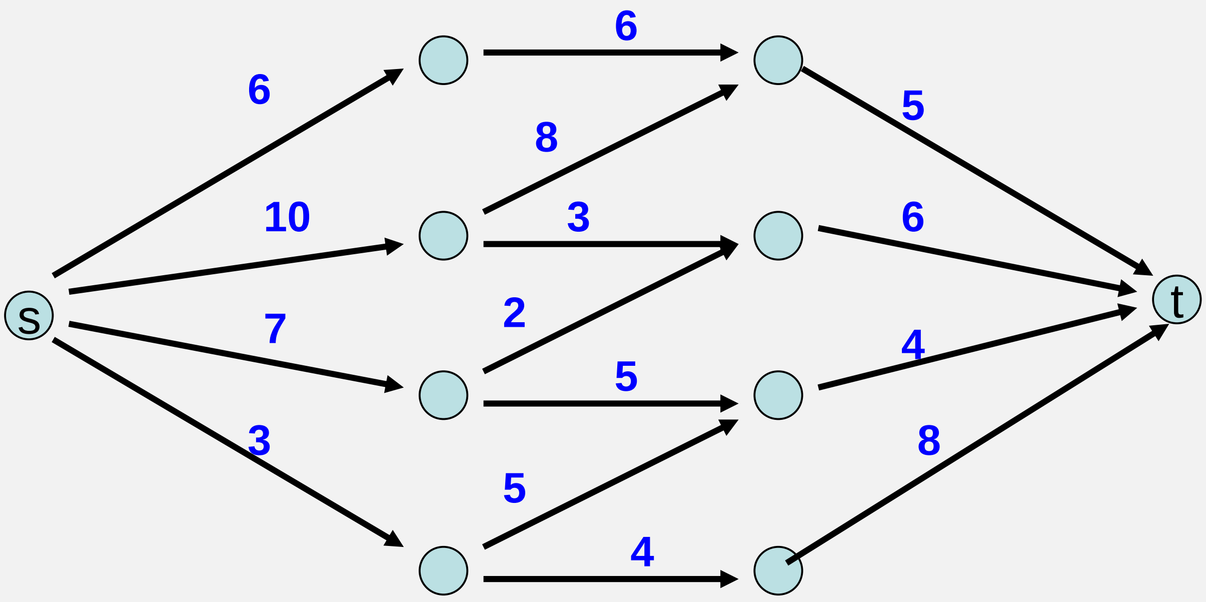


Cap(S,T) = 95, Flow(S,T) = 80 – 15 = 65

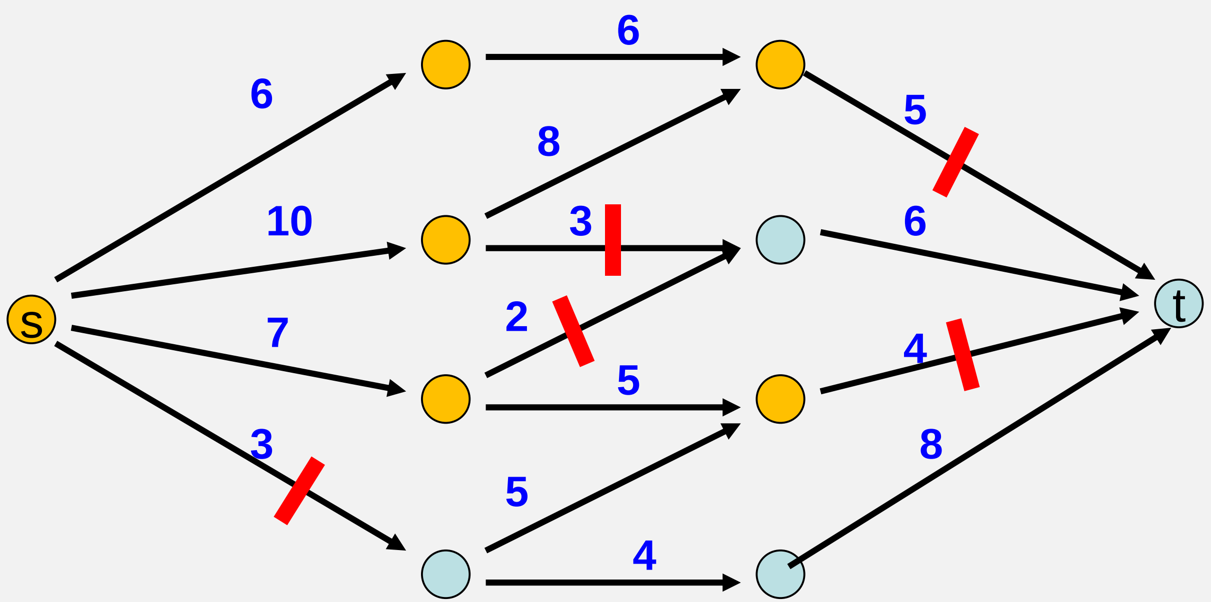
Minimum value cut



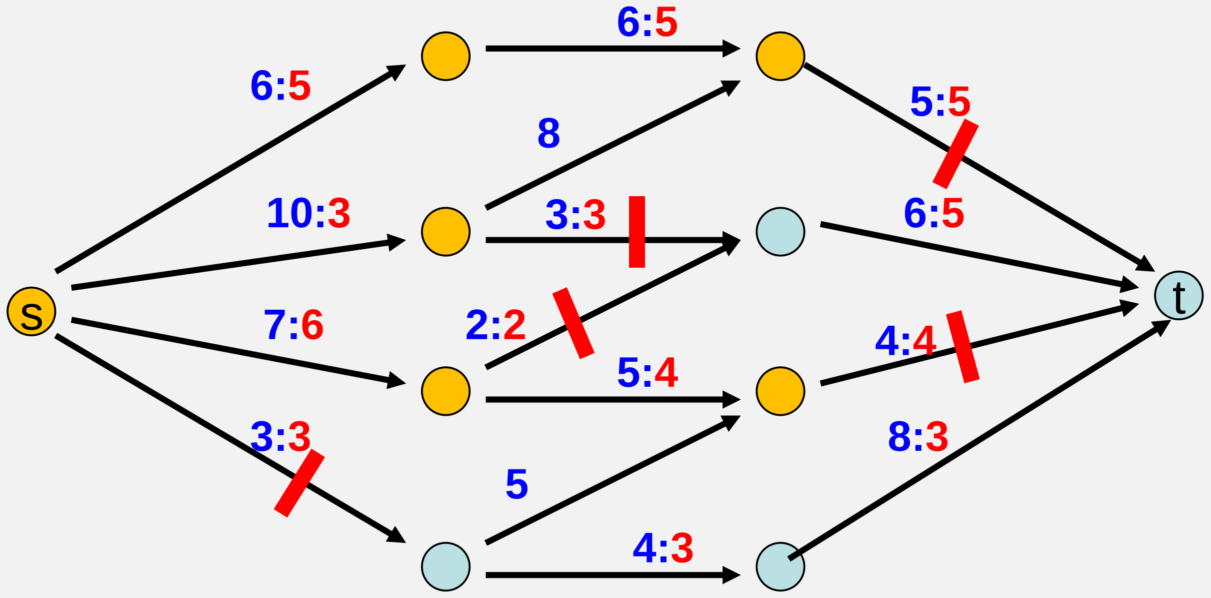
Find a minimum value cut



Find a minimum value cut

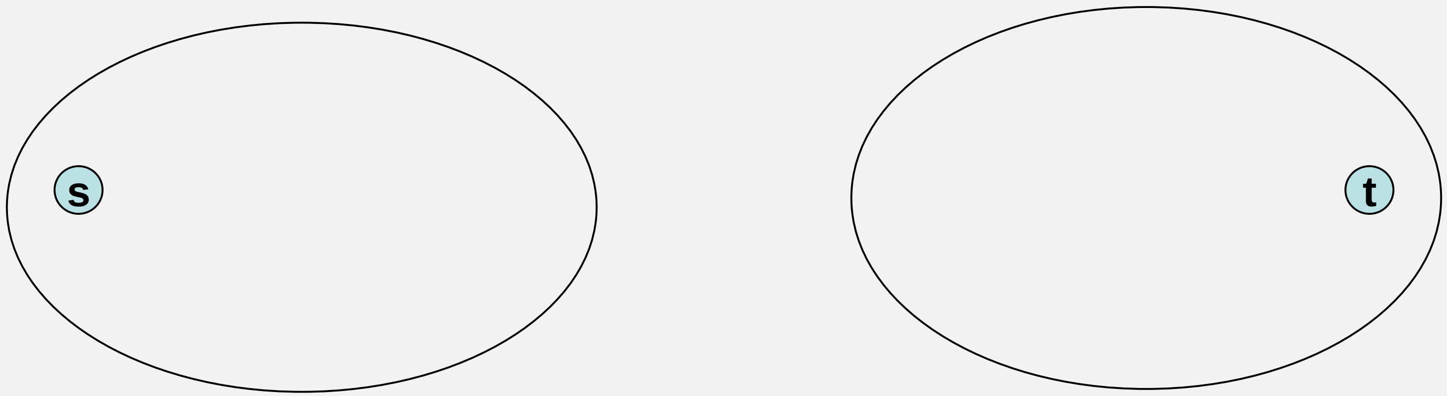


Find a minimum value cut



MaxFlow – MinCut Theorem

- There exists a flow which has the same value of the minimum cut
- Proof: Consider a flow where the residual graph has no s-t path with positive capacity
- Let S be the set of vertices in G_R reachable from s with paths of positive capacity



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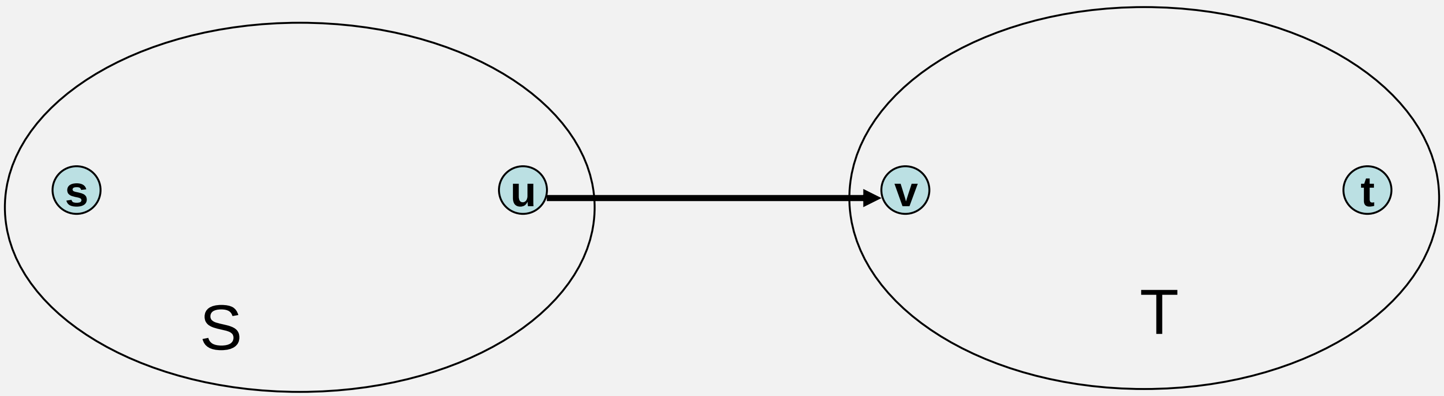


What can we say about the flows and capacity between u and v?

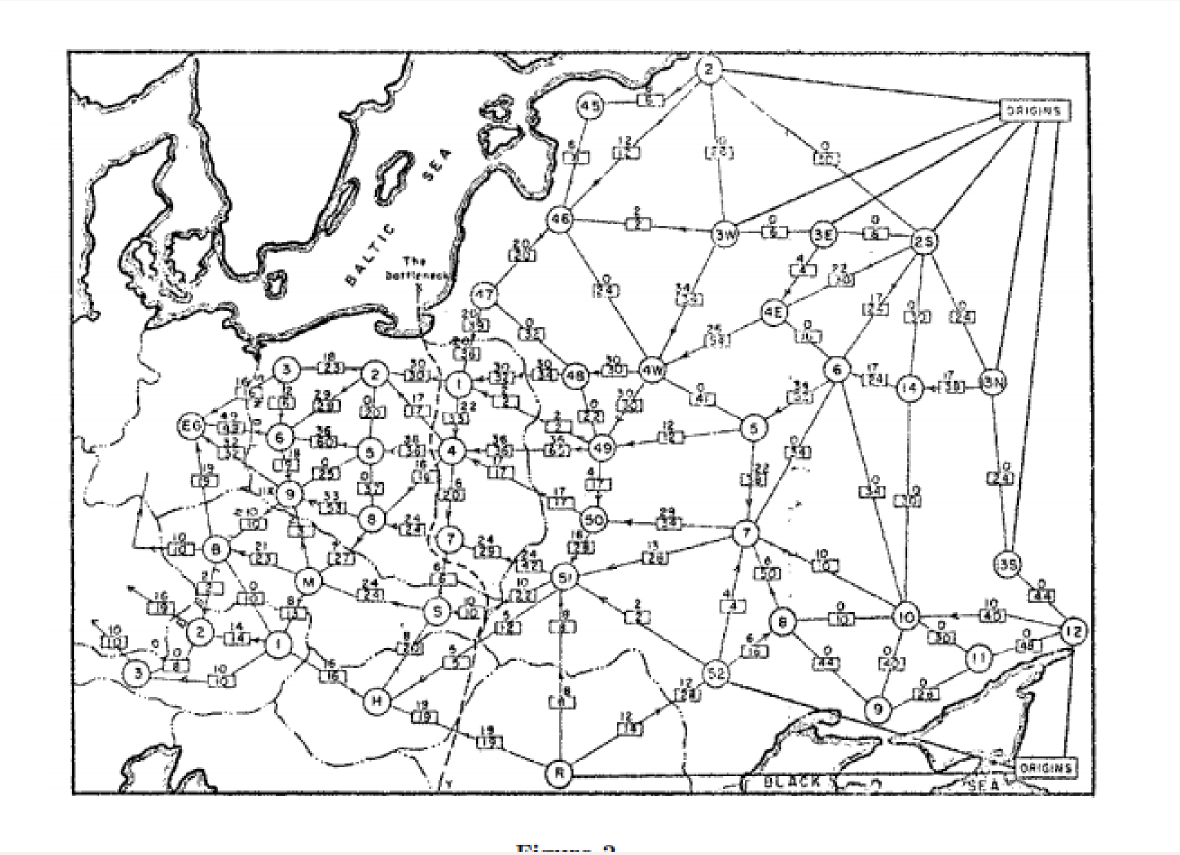
Max Flow - Min Cut Theorem

- Ford-Fulkerson algorithm finds a flow where the residual graph is disconnected, hence FF finds a maximum flow.

- If we want to find a minimum cut, we begin by looking for a maximum flow.

History

- Ford / Fulkerson studied network flow in the context of the Soviet Rail Network



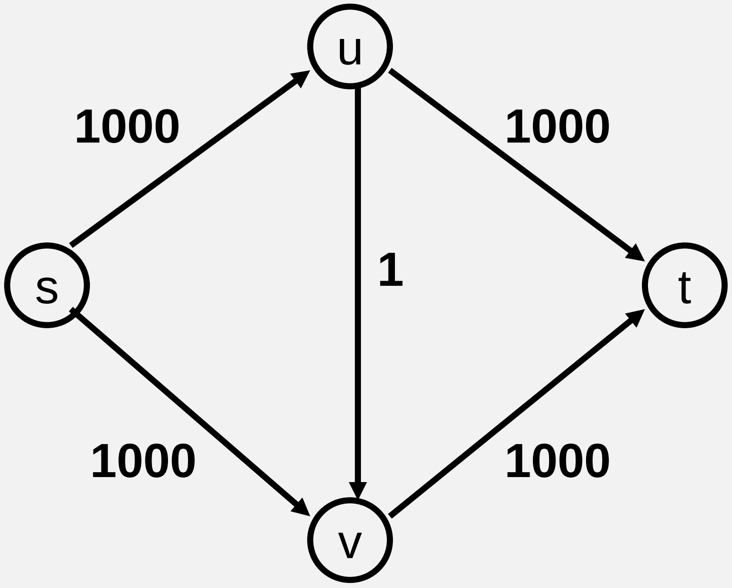
Ford Fulkerson Runtime

- Cost per phase   X   number of phases

- Phases
  - Capacity leaving source: C
  - Add at least one unit per phase
- Cost per phase
  - Build residual graph: O(m)
  - Find s-t path in residual: O(m)

Performance

- The worst case performance of the Ford-Fulkerson algorithm is horrible



Better methods of finding augmenting paths

- Find the maximum capacity augmenting path
  - $O(m^2\log(C))$ time algorithm for network flow
- Find the shortest augmenting path
  - $O(m^2n)$ time algorithm for network flow
- Find a blocking flow in the residual graph
  - $O(mn\log n)$ time algorithm for network flow

Problem Reduction

- Reduce Problem A to Problem B
  - Convert an instance of Problem A to an instance of Problem B
  - Use a solution of Problem B to get a solution to Problem A
- Practical
  - Use a program for Problem B to solve Problem A
- Theoretical
  - Show that Problem B is at least as hard as Problem A

Problem Reduction Examples

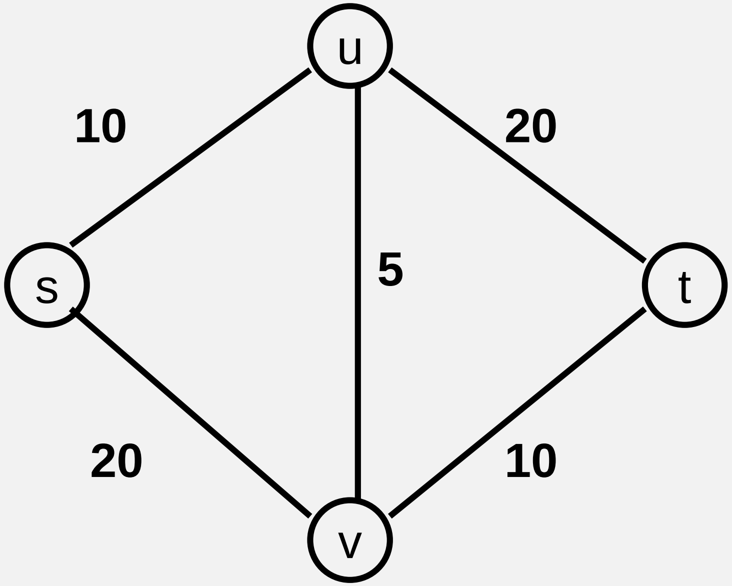
- Reduce the problem of finding the Maximum of a set of integers to finding the Minimum of a set of integers

Find the maximum of: 8, -3, 2, 12, 1, -6

Construct an equivalent minimization problem

Undirected Network Flow

- Undirected graph with edge capacities
- Flow may go either direction along the edges (subject to the capacity constraints)



Construct an equivalent flow problem

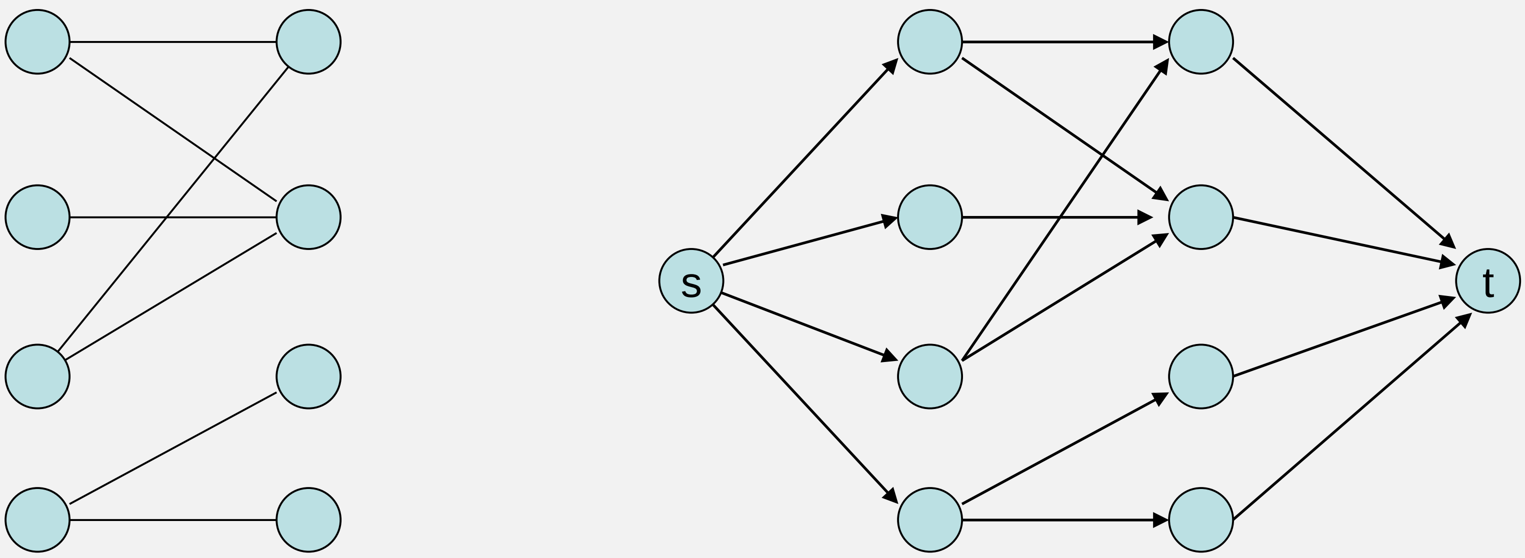
Bipartite Matching

- A graph G=(V,E) is bipartite if the vertices can be partitioned into disjoints sets X,Y
- A matching M is a subset of the edges that does not share any vertices
- Find a matching as large as possible

Application

- A collection of teachers
- A collection of courses
- And a graph showing which teachers can teach which courses

Converting Matching to Network Flow



Multi-source network flow

- Multi-source network flow
  - Sources $s_1, s_2, . . ., s_k$
  - Sinks $t_1, t_2, ... , t_j$
- Solve with Single source network flow

Resource Allocation: Assignment of reviewers

- A set of papers $P_1, \ldots, P_n$
- A set of reviewers $R_1, \ldots, R_m$
- Paper P_i requires A_i reviewers
- Reviewer R_j can review B_j papers
- For each reviewer R_j, there is a list of paper $L_{j1}, \ldots, L_{jk}$ that R_j is qualified to review