Lecture24

CSE 417
Algorithms and Complexity

Autumn 2023
Lecture 24
Network Flow, Part 1

Announcements

- Homework 8, Due Wednesday, Nov 29
- Homework 9, Due Friday, Dec 8

Network Flow

Outline

- Network flow definitions
- Flow examples
- Augmenting Paths
- Residual Graph
- Ford Fulkerson Algorithm
- Cuts
- Maxflow-MinCut Theorem

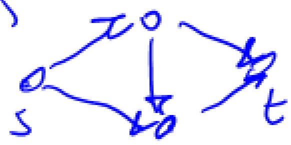
1956

Network Flow Definitions

- Capacity

$0 \le c(e)$

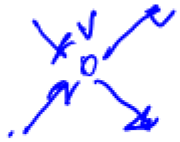
- Source, Sink



- Capacity Condition

$0 \le f(e) \le c(e)$

- Conservation Condition



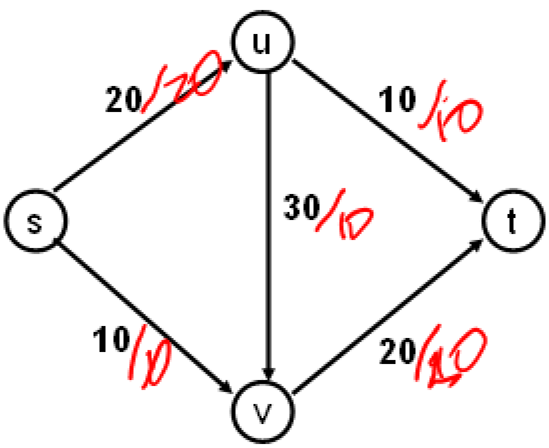
for $v \ne s,t$

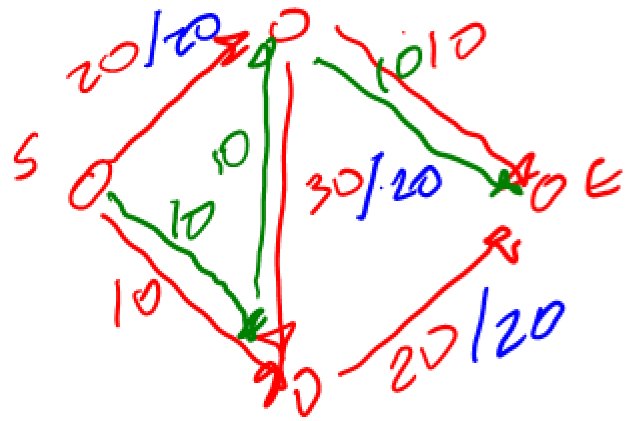
$$\sum_{w \in N^-(v)} f(w,v) = \sum_{x \in N^+(v)} c(v,x)$$

- Value of a flow

$$\sum_{v \in N^+(s)} f(s,v)$$

Flow Example

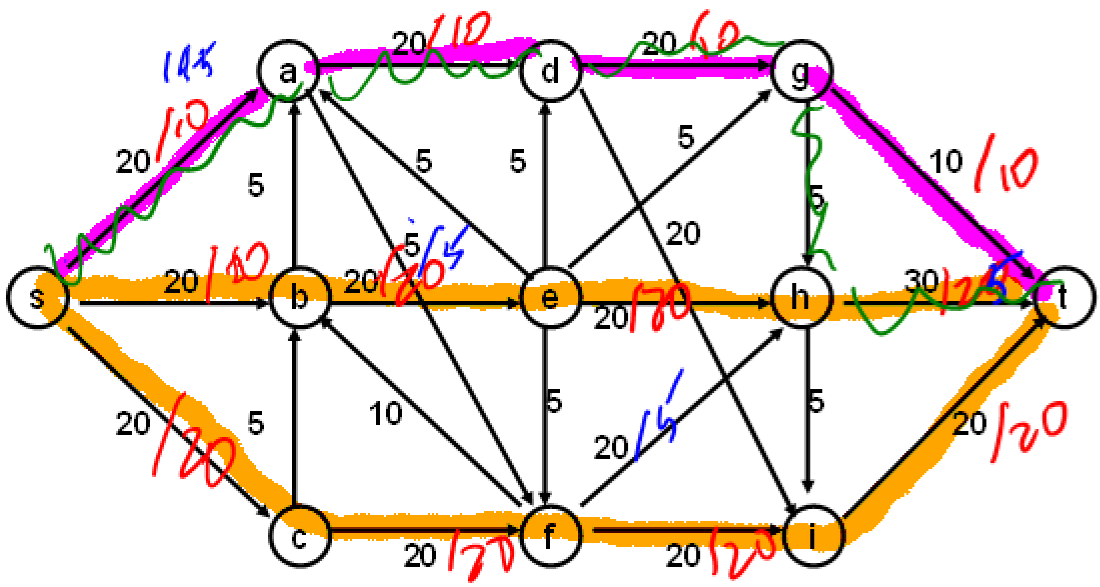




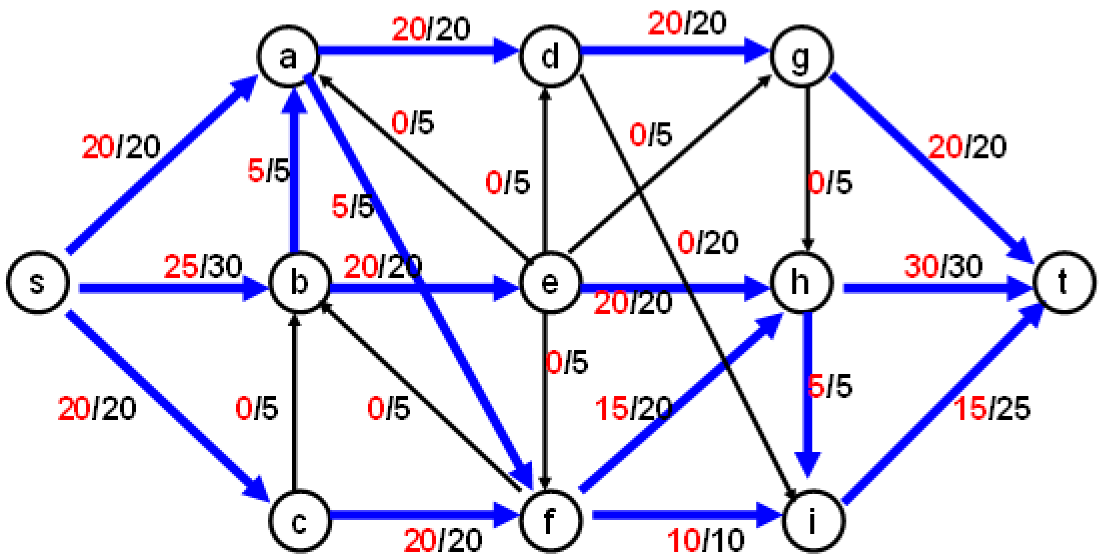
Network Flow Definitions

- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, c(e) >= 0
- Problem, assign flows f(e) to the edges such that:
  - 0 <= f(e) <= c(e)
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is a large as possible

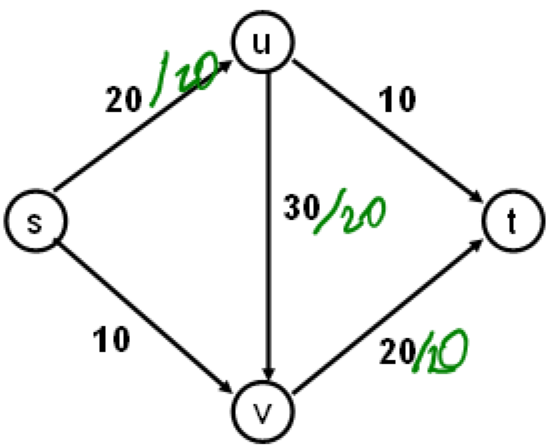
Flow Example

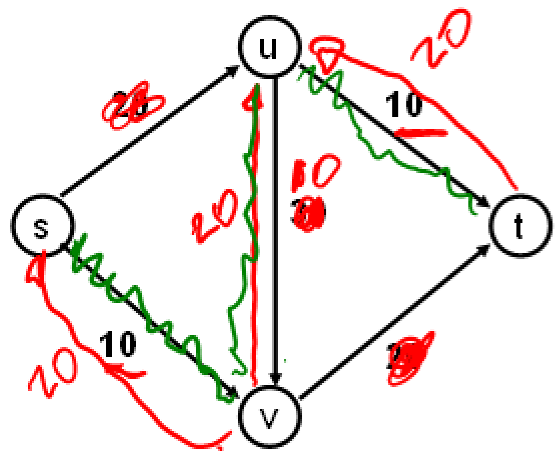


Find a maximum flow



Flow Example

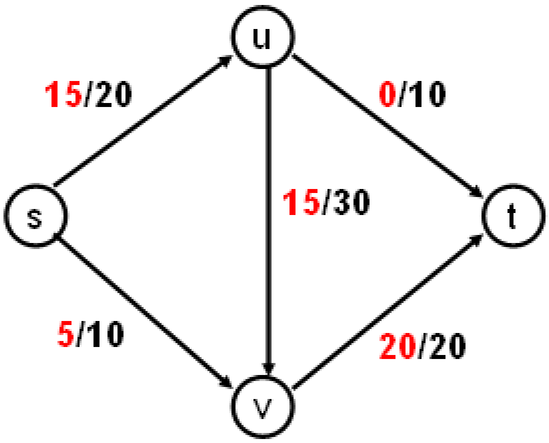


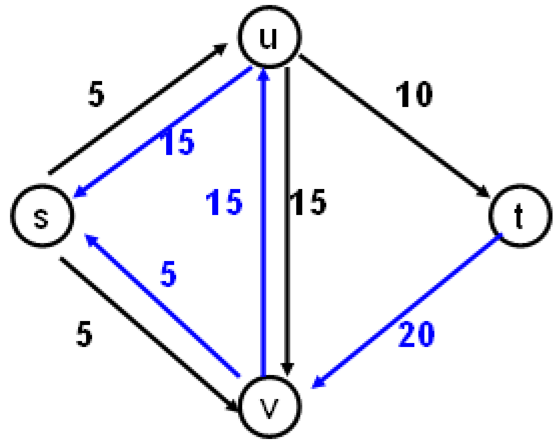


Residual Graph

- Flow graph showing the remaining capacity
- Flow graph G, Residual Graph G_R
  - G: edge e from u to v with capacity c and flow f
  - G_R: edge e' from u to v with capacity c – f
  - G_R: edge e'' from v to u with capacity f

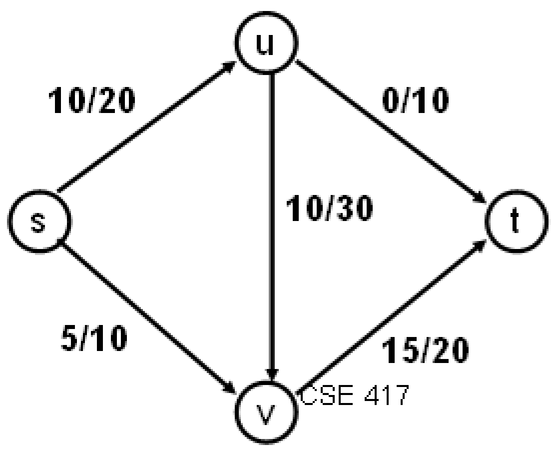
Flow assignment and the residual graph



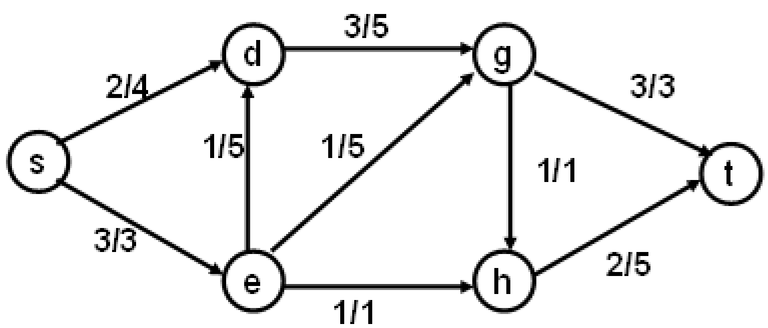


Augmenting Path Algorithm

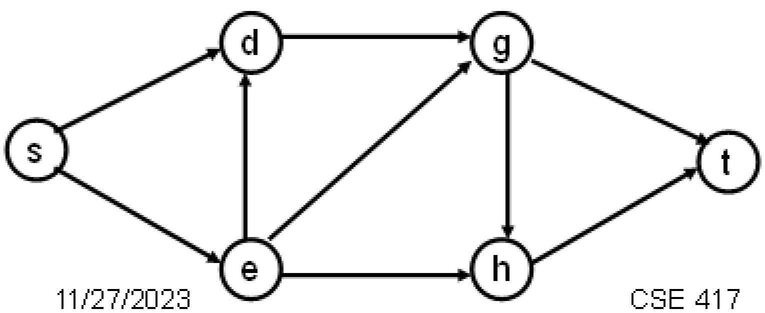
- Augmenting path
  - Vertices $v_1, v_2, \ldots, v_k$
    - $v_1 = s$, $v_k = t$
    - Possible to add b units of flow between v_j and v_{j+1} for j = 1 … k-1



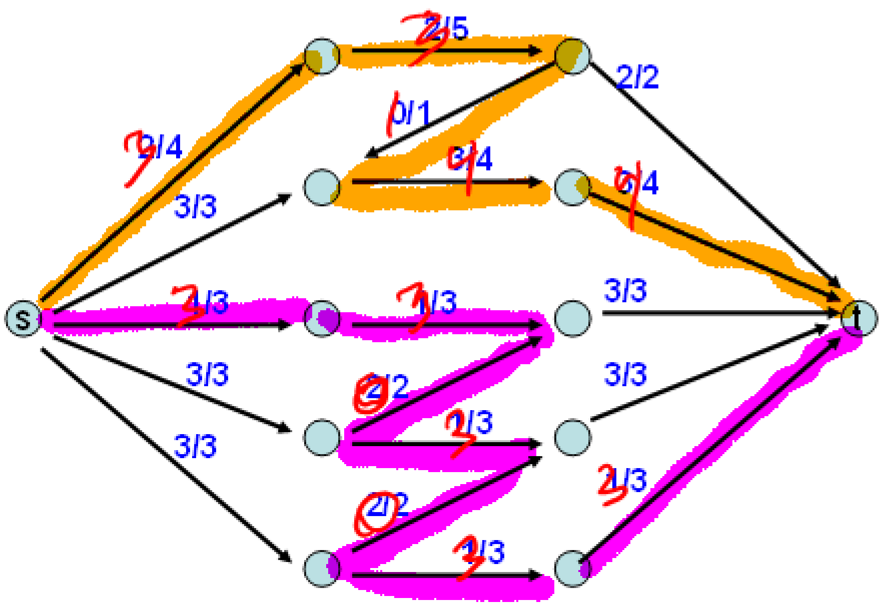
Build the residual graph



Residual graph:

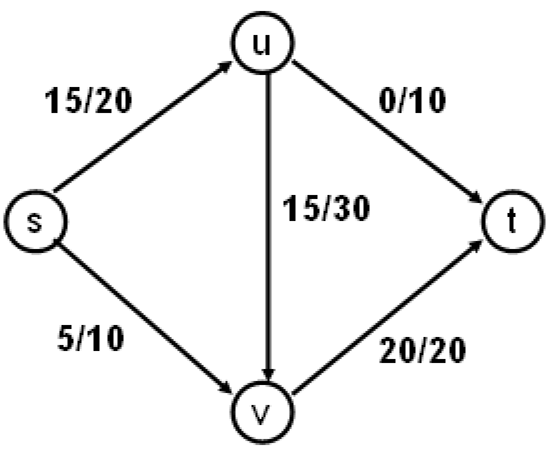


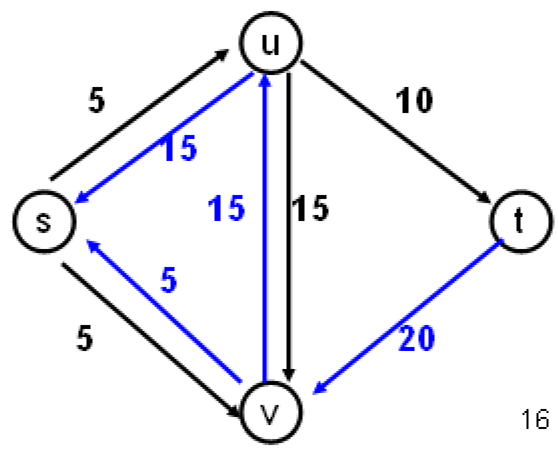
Find two augmenting paths



Augmenting Path Lemma

- Let P = $v_1, v_2, \ldots, v_k$ be a path from s to t with minimum capacity b in the residual graph.
- b units of flow can be added along the path P in the flow graph.





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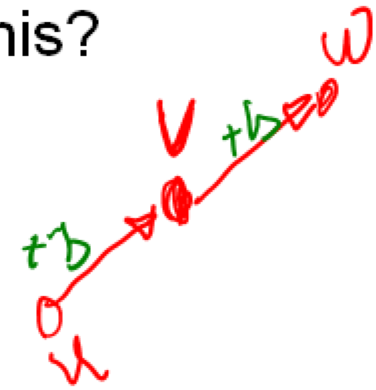
Proof

- Add b units of flow along the path P
- What do we need to verify to show we have a valid flow after we do this?
  - Capacity Restriction
  - Flow Conservation



Ford-Fulkerson Algorithm (1956)

while not done

Construct residual graph G_R

Find an s-t path P in G_R with capacity $b > 0$

Add b units along in G

If the sum of the capacities of edges leaving S is at most C, then the algorithm takes at most C iterations