CSE 417 Algorithms and Complexity

Autumn 2023

Lecture 22

Longest Common Subsequence

Announcements

- Lecture plans
  - Monday: Longest Common Subsequence
  - Wednesday: Shortest Paths
  - Friday: No Class
  - After Thanksgiving: Network Flow + NP Completeness
- Homework plans
  - HW 8, Due Wednesday, November 29
  - HW 9, Due Friday, December 8

Last week, subset sum

- Given integers $\{w_1,\ldots,w_n\}$ and an integer K
- Find a subset that is as large as possible that does not exceed K
- Opt[j, K] the largest subset of $\{w_1, \ldots, w_j\}$ that sums to at most K
- $Opt[j, K] = \max(Opt[j - 1, K], Opt[j - 1, K - w_j] + w_j)$

```
for j = 1 to n
    for k = 1 to W
        Opt[j, k] = max(Opt[j-1, k], Opt[j-1, k-w_j] + w_j)
```

Two dimensional dynamic programming

Subset sum and knapsack

$$Opt[\,j, K] = \max(Opt[\,j - 1, K], Opt[\,j - 1, K - w_j] + w_j)$$

$$Opt[\,j, K] = \max(Opt[\,j - 1, K], Opt[\,j - 1, K - w_j] + v_j)$$

4	0																
3	0																
2	0																
1	0																
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Reducing dimensions

- Computing values in the array only requires the previous row
  - Easy to reduce this to just tracking two rows
  - And sometimes can be implemented in a single row
- Space savings is significant in practice
- Reconstructing values is harder

Longest Common Subsequence

- $C=c_1 \ldots c_g$ is a subsequence of $A=a_1 \ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

ocurranec

occurrence

attacggct

tacgacca

Determine the LCS of the following strings

BARTHOLEMEWSIMPSON

KRUSTYTHECLOWN

String Alignment Problem

- Align sequences with gaps

CAT TGA AT

CAGAT AGGA

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Note: the problem is often expressed as a minimization problem, with $\gamma_{xx} = 0$ and $\delta_x > 0$

Recursive Version

```
LCS(a₁a₂…aₘ, b₁b₂…bₙ){
    if (aₘ == bₙ)
        return LCS(a₁a₂…aₘ₋₁, b₁b₂…bₙ₋₁) + 1;
    else
        return max(LCS(a₁a₂…aₘ₋₁, b₁b₂…bₙ),
                   LCS(a₁a₂…aₘ, b₁b₂…bₙ₋₁);
}
```

LCS Optimization

- $A = a_1a_2\ldots a_m$
- $B = b_1b_2\ldots b_n$

- $Opt[\,j, k]$ is the length of $LCS(a_1a_2\ldots a_j, b_1b_2\ldots b_k)$

Optimization recurrence

If $a_j = b_k$, Opt[j,k] = 1 + Opt[j-1, k-1]

If $a_j \neq b_k$, Opt[j,k] = max(Opt[j-1,k], Opt[j,k-1])

Give the Optimization Recurrence for the String Alignment Problem

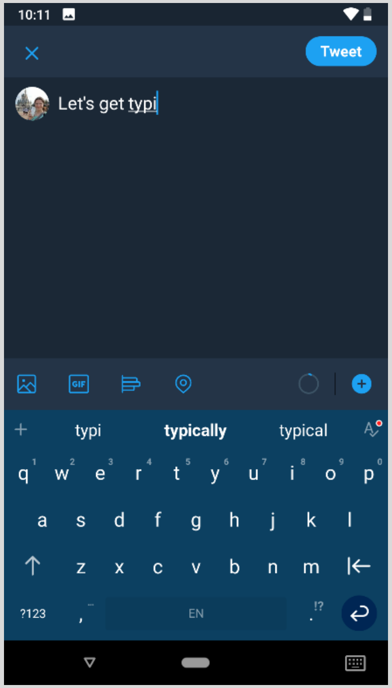
- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Opt[j, k] =

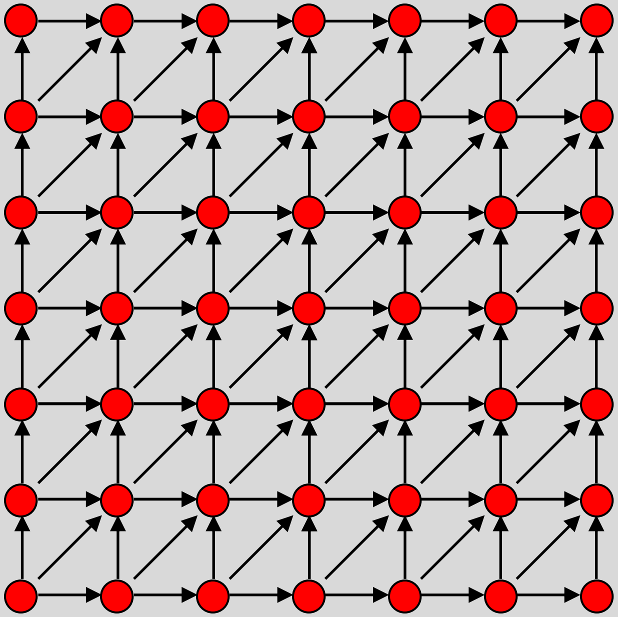
Let a_j = x and b_k = y
Express as minimization

String edit with Typo Distance

- Find closest dictionary word to typed word
- Dist(‘a’, ‘s’) = 1
- Dist(‘a’, ‘u’) = 6
- Capture the likelihood of mistyping characters



Dynamic Programming Computation

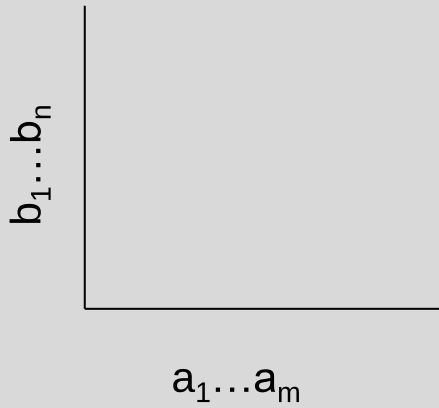


Code to compute Opt[n, m]

```
for (int i = 0; i < n; i++)
  for (int j = 0; j < m; j++)
    if (A[ i ] == B[ j ] )
      Opt[ i,j ] = Opt[ i-1, j-1 ] + 1;
    else if (Opt[ i-1, j ] >= Opt[ i, j-1 ])
       Opt[ i, j ] := Opt[ i-1, j ];
    else
       Opt[ i, j ] := Opt[ i, j-1];
```

Storing the path information

```
A[1..m],  B[1..n]
for i := 1 to m     Opt[i, 0] := 0;
for j := 1 to n     Opt[0,j] := 0;
Opt[0,0] := 0;
for i := 1 to m
        for j := 1 to n
                if A[i] = B[j]  {  Opt[i,j] := 1 + Opt[i-1,j-1];  Best[i,j] := Diag; }
                else if Opt[i-1, j] >= Opt[i, j-1]
                        {  Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }
                else      {  Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }
```



Reconstructing Path from Distances

```
LCS Arguments
  211031321102033212120000321302
  100222010121310130323233121011
_0 _0 _0 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1
_1 _1 _1 _1 _1 _1 _1 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2
_1 _1 _1 _1 _1 _1 _1 _2 _2 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3
_1 _2 _2 _2 _2 _2 _2 _2 _3 _3 _3 _3 _3 _3 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4
_1 _2 _2 _2 _2 _2 _2 _2 _3 _3 _3 _3 _4 _4 _4 _4 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5
_1 _2 _2 _2 _2 _2 _2 _3 _3 _4 _4 _4 _4 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _6 _6 _6 _6 _6 _6
_1 _2 _2 _2 _2 _2 _2 _3 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6
_1 _2 _2 _3 _3 _3 _3 _3 _3 _4 _5 _5 _5 _5 _5 _5 _6 _6 _6 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7
_1 _2 _2 _3 _3 _3 _3 _4 _4 _4 _5 _6 _6 _6 _6 _6 _6 _6 _6 _7 _7 _7 _7 _7 _8 _8 _8 _8 _8 _8
_1 _2 _2 _3 _3 _3 _3 _4 _4 _5 _5 _6 _6 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _8 _8 _9 _9 _9 _9
_1 _2 _3 _3 _3 _3 _4 _4 _5 _5 _5 _6 _6 _7 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _9 10 10 10
_1 _2 _3 _4 _4 _4 _4 _4 _5 _5 _6 _6 _6 _7 _8 _8 _8 _8 _8 _9 _9 _9 _9 _9 _9 _9 _9 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _6 _7 _8 _8 _8 _9 _9 _9 _9 _9 _9 _9 _9 _9 _9 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 10 10 10 10 10 10 10 10 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 10 11 11 11 11 11 11 11 11 11 11
_1 _2 _3 _4 _5 _5 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 11 11 12 12 12 12 12 12 12 12 12
_1 _2 _3 _4 _5 _5 _5 _6 _6 _6 _6 _7 _7 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 13 13 13 13 13
_1 _2 _3 _4 _5 _6 _6 _6 _6 _6 _7 _7 _7 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 14 14 14 14
_1 _2 _3 _4 _5 _6 _6 _7 _7 _7 _7 _8 _8 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 15 15 15 15
_1 _2 _3 _4 _5 _6 _6 _7 _7 _7 _8 _8 _8 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 15 15 15 15
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _9 _9 _9 _9 10 10 11 11 12 12 13 13 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _9 _9 _9 _9 _9 _9 10 10 11 12 12 13 13 13 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _8 _8 _9 _9 10 10 10 10 10 10 10 11 12 12 13 13 13 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _8 _9 _9 10 11 11 11 11 11 11 11 12 13 13 14 14 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _9 _9 _9 10 11 11 12 12 12 12 12 12 13 13 14 14 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _9 _9 10 10 11 11 12 12 12 12 12 13 13 14 14 14 14 15 15 16 17 17
```

How good is this algorithm?

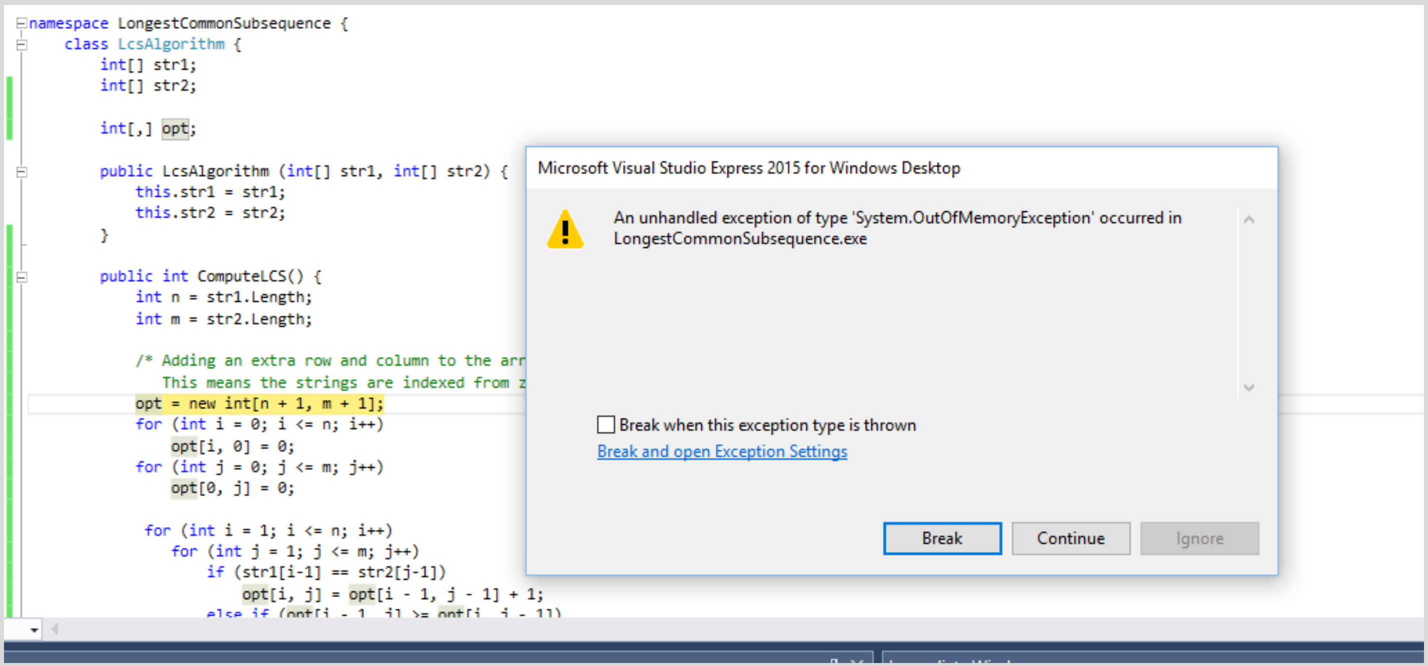
- Is it feasible to compute the LCS of two strings of length 300,000 on a standard desktop PC? Why or why not.

Implementation 1

```
public int ComputeLCS() {
            int n = str1.Length;
            int m = str2.Length;

            int[,] opt = new int[n + 1, m + 1];
            for (int i = 0; i <= n; i++)
                opt[i, 0] = 0;
            for (int j = 0; j <= m; j++)
                opt[0, j] = 0;

             for (int i = 1; i <= n; i++)
                for (int j = 1; j <= m; j++)
                    if (str1[i-1] == str2[j-1])
                        opt[i, j] = opt[i - 1, j - 1] + 1;
                    else if (opt[i - 1, j] >= opt[i, j - 1])
                        opt[i, j] = opt[i - 1, j];
                    else
                        opt[i, j] = opt[i, j - 1];

            return opt[n,m];
        }
```

N = 17000

Runtime should be about 5 seconds*



* Personal PC, 10 years old

Manufacturer:	Dell
Model:	Optiplex 990
Processor:	Intel(R) Core(TM) i5-2400 CPU @ 3.10GHz 3.10 GHz
Installed memory (RAM):	8.00 GB (7.88 GB usable)
System type:	64-bit Operating System, x64-based processor

Implementation 2

```
public int SpaceEfficientLCS() {
            int n = str1.Length;
            int m = str2.Length;
            int[] prevRow = new int[m + 1];
            int[] currRow = new int[m + 1];

            for (int j = 0; j <= m; j++)
                prevRow[j] = 0;

            for (int i = 1; i <= n; i++) {
                currRow[0] = 0;
                for (int j = 1; j <= m; j++) {
                    if (str1[i - 1] == str2[j - 1])
                        currRow[j] = prevRow[j - 1] + 1;
                    else if (prevRow[j] >= currRow[j - 1])
                        currRow[j] = prevRow[j];
                    else
                        currRow[j] = currRow[j - 1];
                }
                for (int j = 1; j <= m; j++)
                    prevRow[j] = currRow[j];
            }

            return currRow[m];
        }
```

N = 300000

N: 10000 Base 2 Length: 8096	Gamma:	0.8096	Runtime:00:00:01.86
N: 20000 Base 2 Length: 16231	Gamma:	0.81155	Runtime:00:00:07.45
N: 30000 Base 2 Length: 24317	Gamma:	0.8105667	Runtime:00:00:16.82
N: 40000 Base 2 Length: 32510	Gamma:	0.81275	Runtime:00:00:29.84
N: 50000 Base 2 Length: 40563	Gamma:	0.81126	Runtime:00:00:46.78
N: 60000 Base 2 Length: 48700	Gamma:	0.8116667	Runtime:00:01:08.06
N: 70000 Base 2 Length: 56824	Gamma:	0.8117715	Runtime:00:01:33.36

N: 300000 Base 2 Length: 243605 Gamma: 0.8120167 Runtime:00:28:07.32

Observations about the Algorithm

- The computation can be done in O(m+n) space if we only need one column of the Opt values or Best Values

- The computation requires O(nm) space if we store all of the string information

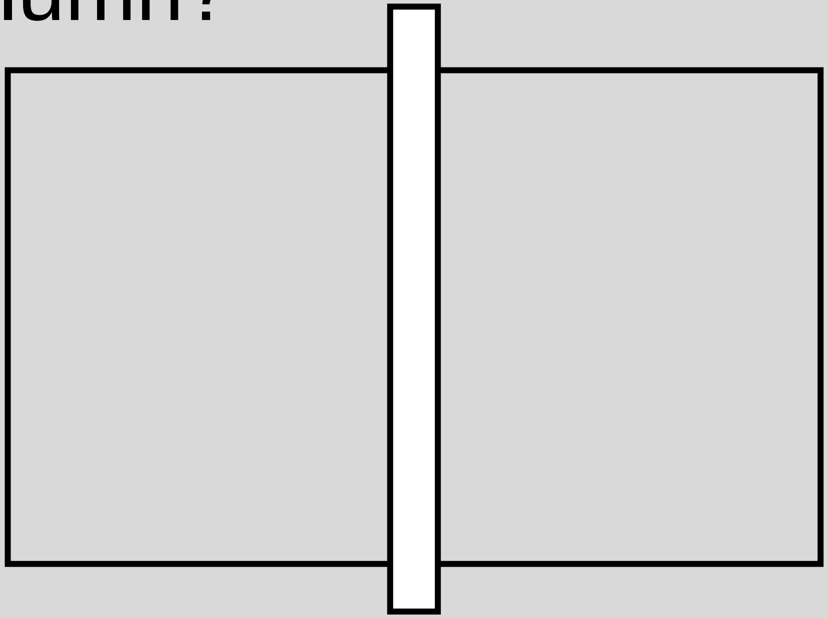
Computing LCS in O(nm) time and O(n+m) space

- Divide and conquer algorithm
- Recomputing values used to save space

- Section 6.7 of the text, but we will not have time to cover in detail (so you are not responsible for section 6.7)

Divide and Conquer Algorithm

- Where does the best path cross the middle column?



- For a fixed i, and for each j, compute the LCS that has a_i matched with b_j

Algorithm Analysis

- $T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm$
- Solution: $T(m,n) <= 2cnm$