CSE 417 Algorithms and Complexity

Lecture 19, Autumn 2023

Dynamic Programming

Announcements

- Dynamic Programming Reading:
  - 6.1-6.2, Weighted Interval Scheduling
  - 6.4 Knapsack and Subset Sum
  - 6.6 String Alignment
    - 6.7* String Alignment in linear space
  - 6.8 Shortest Paths (again)
  - 6.9 Negative cost cycles
    - How to make an infinite amount of money
- Homework 7

Dynamic Programming

- The most important algorithmic technique covered in CSE 417
- Key ideas
  - Express solution in terms of a polynomial number of sub problems
  - Order sub problems to avoid recomputation

Recursion vs Iteration

```
Factorial(n){
    if (n <= 1)
        return 1;
    else
        return n*Factorial(n-1);
}
```

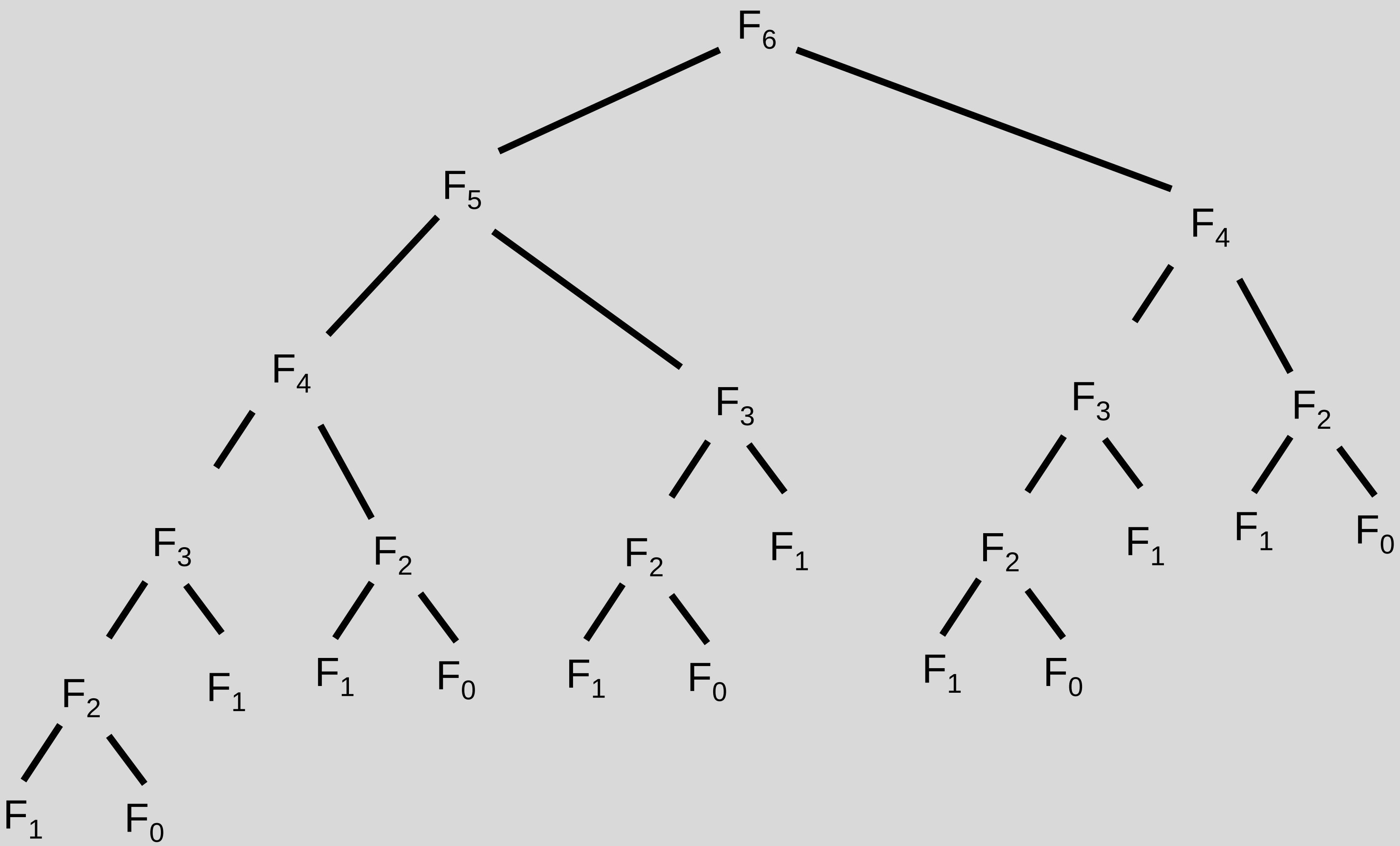
```
Factorial(n){
    v = 1;
    for (i = 2; i <= n; i++)
        v = v*i
    return v;
}
```

Counting Rabbits

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, . . .

$F_0 = 0; \quad F_1 = 1; \quad F_n = F_{n-1} + F_{n-2}$

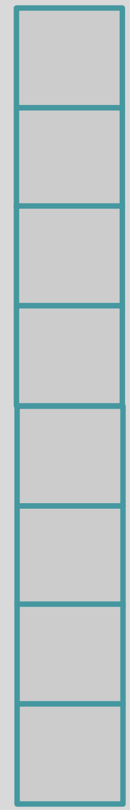
```
Fib(n){
    if (n =  0)
         return 0;
    else if (n = 1)
         return 1;
    else
         return Fib(n-1) + Fib(n-2);
}
```



F_6
F_5
F_4
F_4
F_3
F_3
F_2
F_3
F_2
F_2
F_1
F_2
F_1
F_1
F_0
F_2
F_1
F_1
F_0
F_1
F_0
F_1
F_0
F_1
F_0

Fibonacci with Memoization

```
Fib(n){
    if (n =  0)
        return 0;
    else if (n = 1)
        return 1;
    else
        return Fib(n-1) + Fib(n-2);
}
```



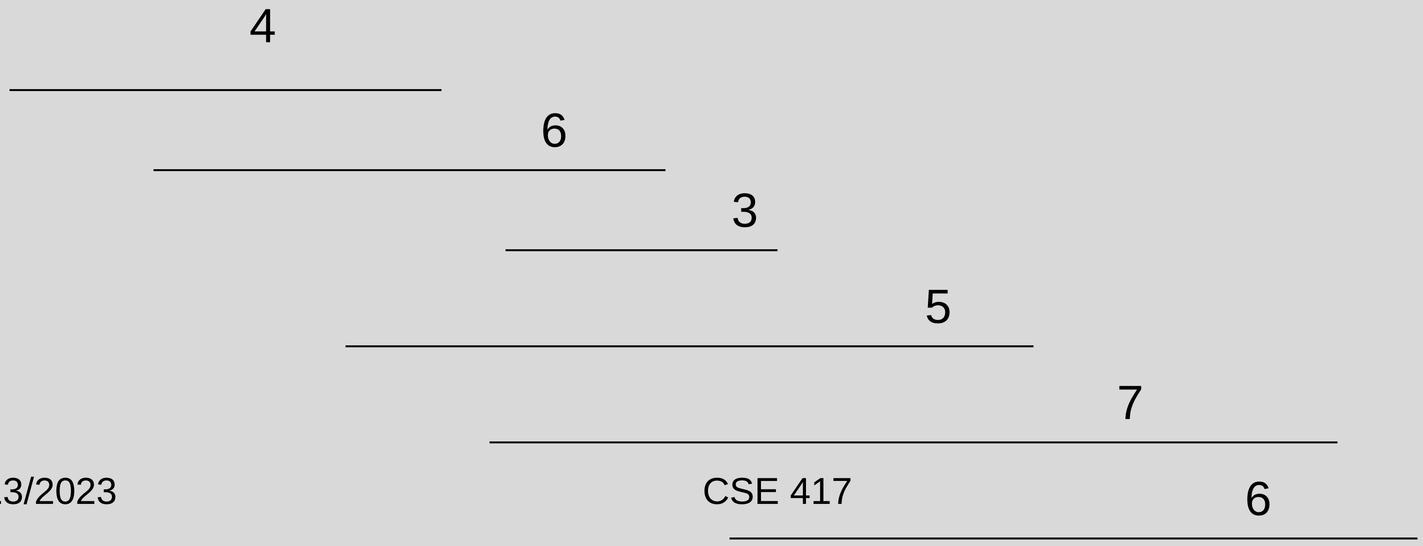
Reordering computation

```
Fib(n){
    int[ ] F = new [n+1]

    F[0] = 0;
    F[1] = 1;
    for (i = 2; i <= n; i++)
        F[i] = F[i-1] + F[i-2];
    return F[n];
}
```

Intervals sorted by end time

Dynamic Programming

- Weighted Interval Scheduling
- Given a collection of intervals $I_1,\ldots,I_n$ with weights $w_1,\ldots,w_n$, choose a maximum weight set of non-overlapping intervals



Intervals sorted by end time

Optimality Condition

- Opt[j] is the maximum weight independent set of intervals I_1, I_2, . . ., I_j
- Opt[j] = max(Opt[j – 1], w_j + Opt[p[j]])
  - Where p[j] is the index of the last interval which finishes before I_j starts

Algorithm

```
MaxValue(j) =
   if j = 0 return 0
   else
        return max( MaxValue(j-1),
                        w_j + MaxValue(p[ j ]))
```

Worst case run time: 2^n

A better algorithm

M[j] initialized to -1 before the first recursive call for all j

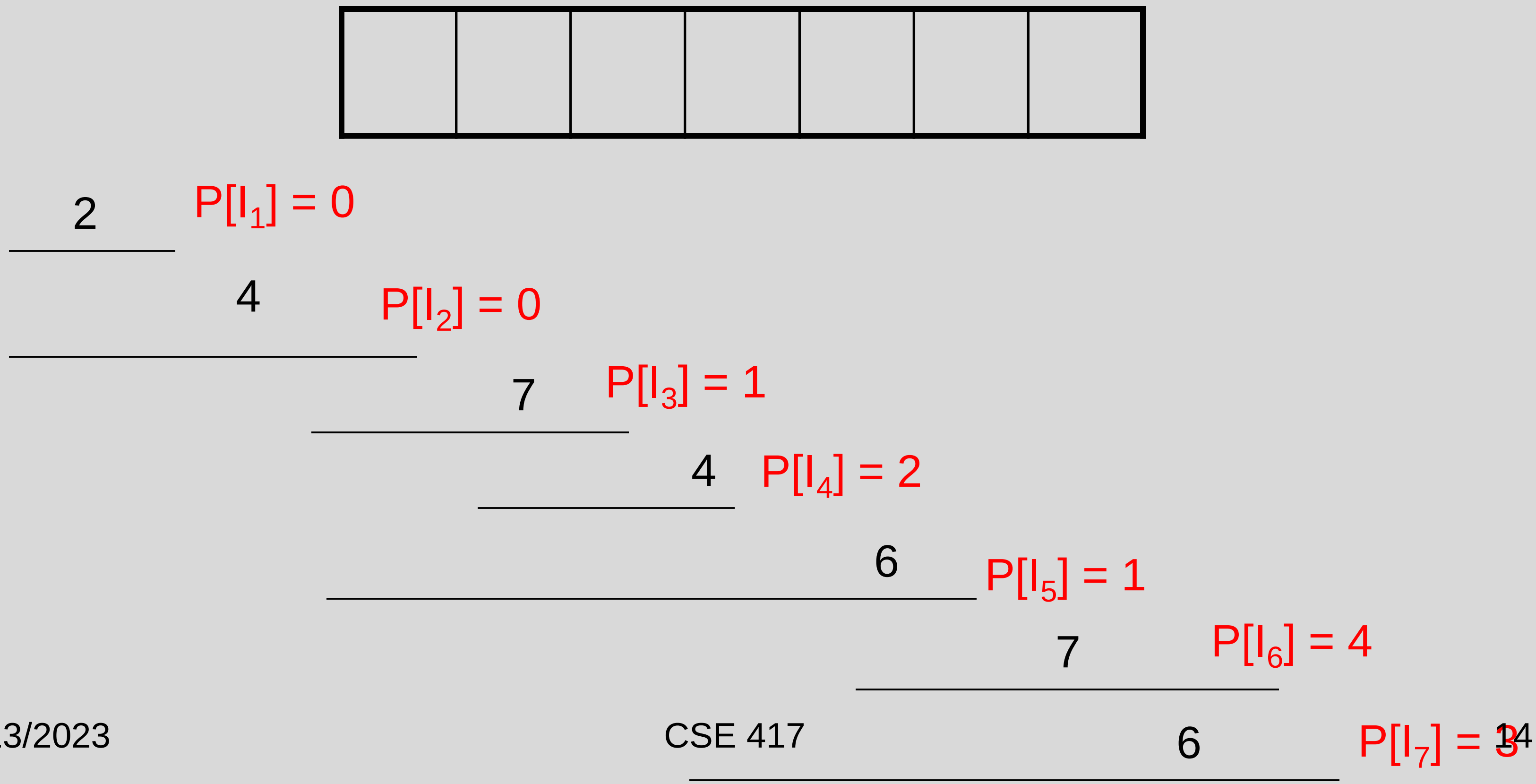
```
MaxValue(j) =
    if j = 0 return 0;
    else if M[ j ] != -1 return M[ j ];
    else
        M[ j ] = max(MaxValue(j-1), w_j + MaxValue(p[ j ]));
        return M[ j ];
```

Iterative Algorithm

```
MaxValue(n){
    int[ ] M = new int[n+1];

    M[0] = 0;

    for (int i = 1; i <= n; i++){
        M[ j ] = max(M[j-1], w_j + M[p[ j ]]);

    return M[n];
}
```

Fill in the array with the Opt values

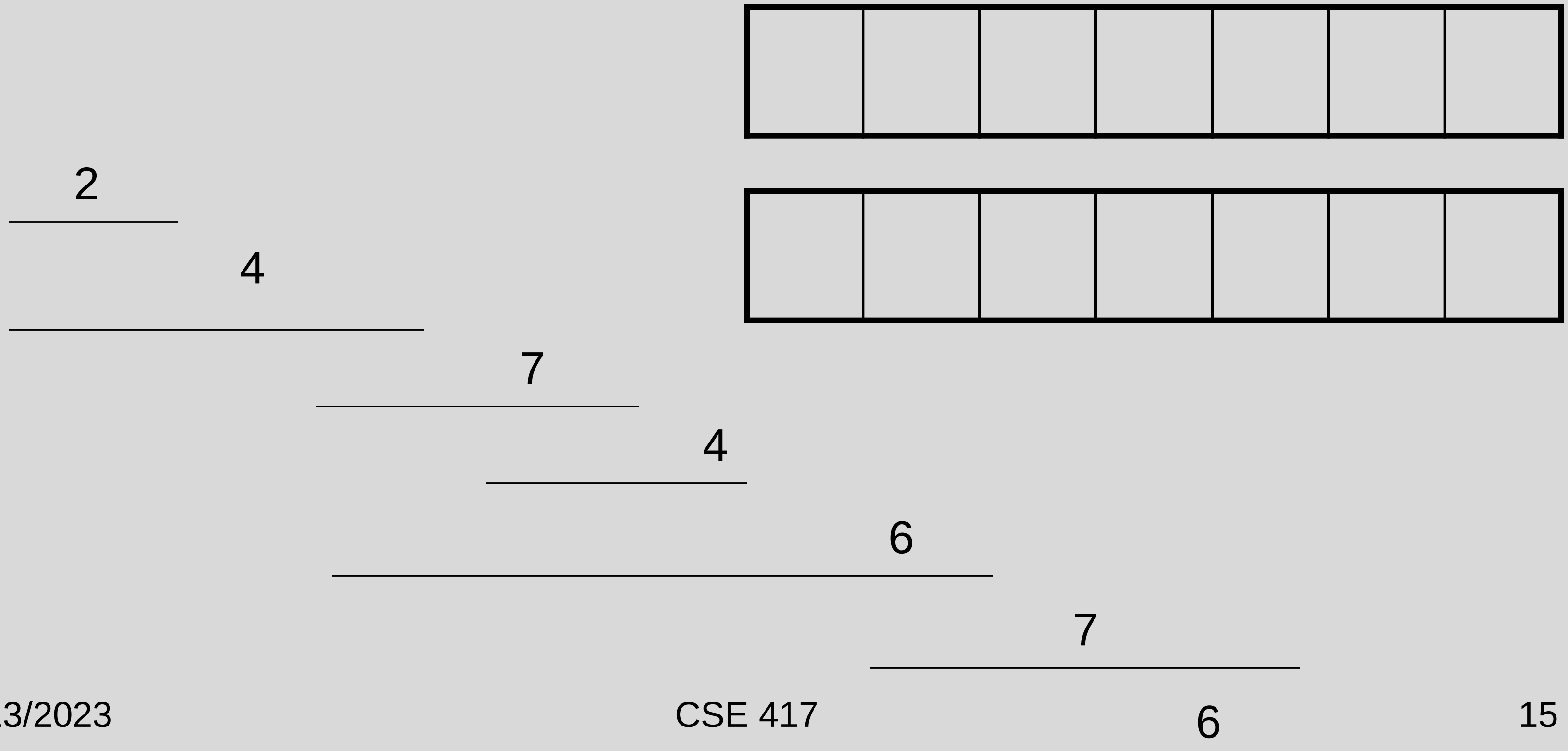
$$Opt[\, j\,] = \max\,(Opt[\, j - 1], w_j + Opt[\, p[\, j\,]\,])$$



Computing the solution

Opt[j] = max (Opt[j – 1], w_j + Opt[p[j]])

Record which case is used in Opt computation



Iterative Algorithm

```
int[] M = new int[n+1];
char[] R = new char[n+1];

M[0] = 0;
for (int j = 1; j < n+1; j++){
        v1 = M[j-1];
        v2 = W[j] + M[P[j]];
        if (v1 > v2) {
                M[j] = v1;
                R[j] = ‘A’;
        }
        else {
                M[j] = v2;
                R[j] = ‘B’;
        }
}
```

Algorithm Summary

- $O(n^2)$ time algorithm for finding maximum weight independent set of intervals
- Key idea: Creating an Opt function to express optimal set of $I_1, I_2, \ldots I_k$ in terms of optimal set of $I_1, I_2, \ldots I_{k-1}$
- Organize computation to avoid recomputation