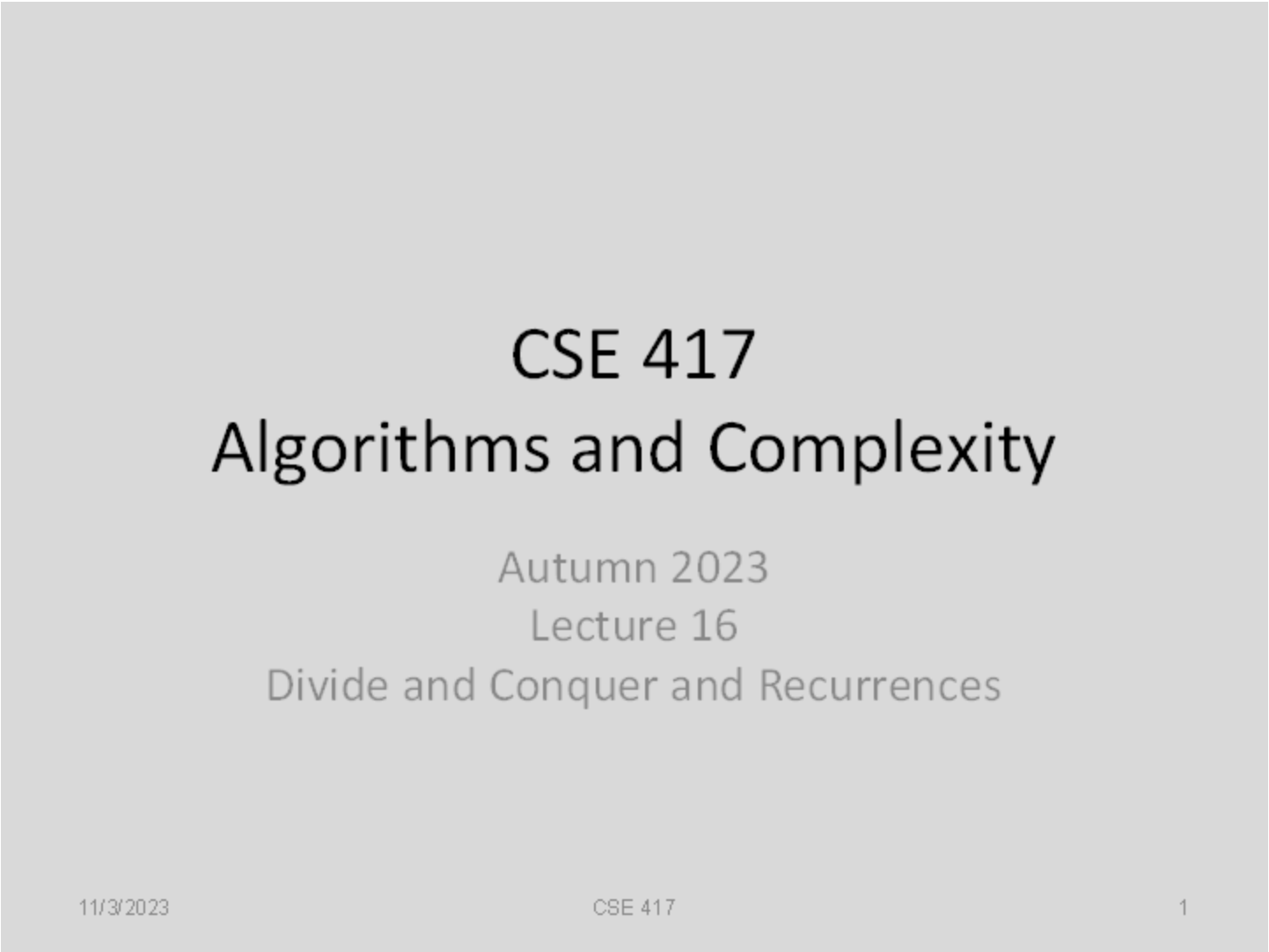
Lecture16


CSE 417
Algorithms and Complexity
Autumn 2023
Lecture 16
Divide and Conquer and Recurrences
11/3/2023
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Divide and Conquer

- Recurrences, Sections 5.1 and 5.2
- Algorithms
  - Median (Selection)
  - Fast Matrix Multiplication
  - Counting Inversions (5.3)
  - Multiplication (5.5)

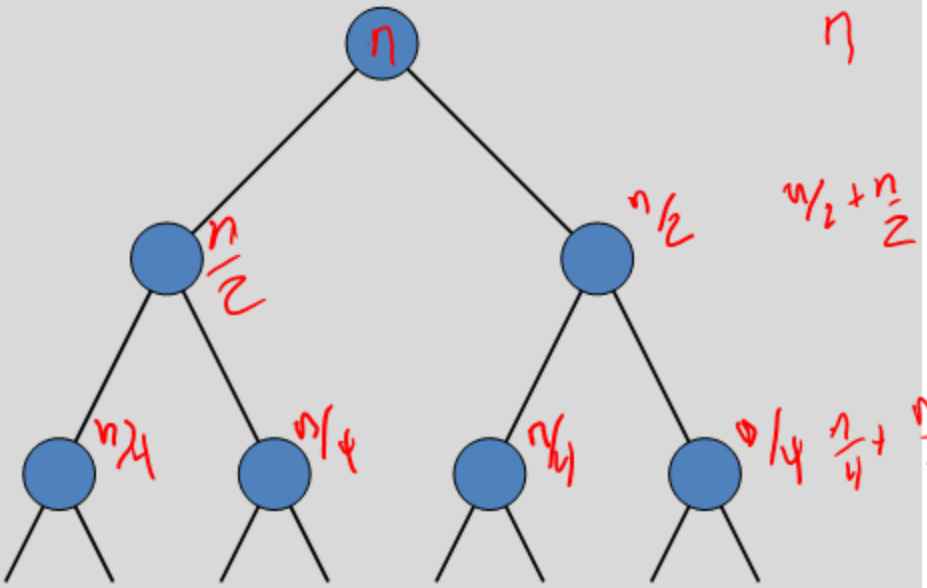
Divide and Conquer : Merge Sort

```
Array MSort(Array a, int n){
   if (n <= 1)  return a;
   return Merge(MSort(a[0 .. n/2], n/2),  MSort(a[n/2+1 .. n-1], n/2);
}
```

$$T(n) = 2T(n/2) + n;\ T(1) = 1;$$

Total work is $O(n(\log n+1))$

Unrolling the recurrence



n

$n/2 + n/2$

$n/4 + n/4 + n/4 + n/4$

n

n

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T(n) = 2T(n/2) + n; T(1) = 1;

Guess & Verify

Substitution

Prove $T(n) \leq n (\log_2 n + 1)$ for $n \geq 1$

Induction – Show P(1) and $\forall_{k<n} P(k) \implies P(n)$

Base Case: $T(1) = 1 = 1 (\log_2 1 + 1)$

Induction: Assume $T(n/2) \leq n/2 (\log_2(n/2) + 1)$

$T(n) = 2\,T(n/2) + n$

$\leq 2\, n/2 (\log_2(n/2) + 1) + n$

$= n (\log_2 n - 1 + 1) + n$

$= n (\log_2 n + 1)$

$T(n) = aT(n/b) + n^c$

Master Theorem

If $T(n) = aT(n/b) + O(n^d)$ for constants $a > 0$, $b > 1$, $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

Binary search

$T(n) = T(\frac{n}{2}) + 1$

$O(\log n)$

$$T(n) = T(n/2) + cn$$

Where does this recurrence arise?

How many games in a tournament starting with n teams

$$T(n) = \frac{n}{2} + T\left(\frac{n}{2}\right)$$

$$T(2) = 1$$

$T(n) = T(\frac{n}{2}) + n$

Solving the recurrence exactly

$T(n) = n + T(\frac{n}{2})$

$= n + \frac{n}{2} + T(\frac{n}{4})$

$= n + \frac{n}{2} + \frac{n}{4} + T(\frac{n}{8}) \quad = 2n-1$

$T(n) = n + T(cn) \qquad c < 1$

$= n + cn + c^2 n + c^3 n + \cdots$

$= n \sum_{i=0}^{p} c^i \quad = n \frac{1-c^{j+1}}{1-c} \quad < n \frac{1}{1-c}$

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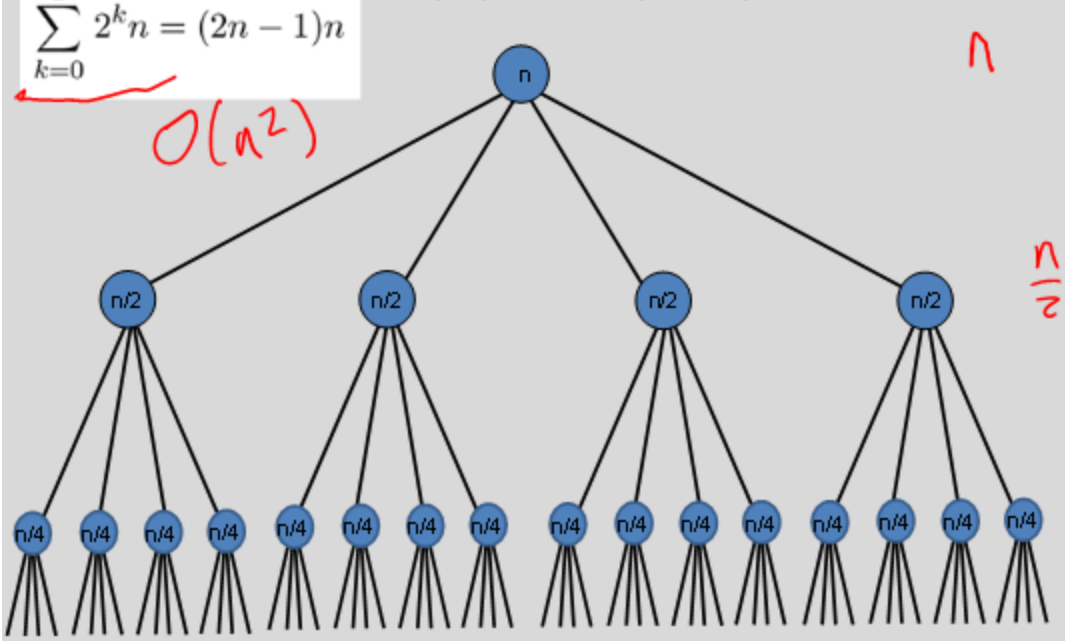
Total Work

$$\sum_{k=0}^{\log n} 2^k n = (2n-1)n$$

$O(n^2)$

$T(n) = 4T(n/2) + n$



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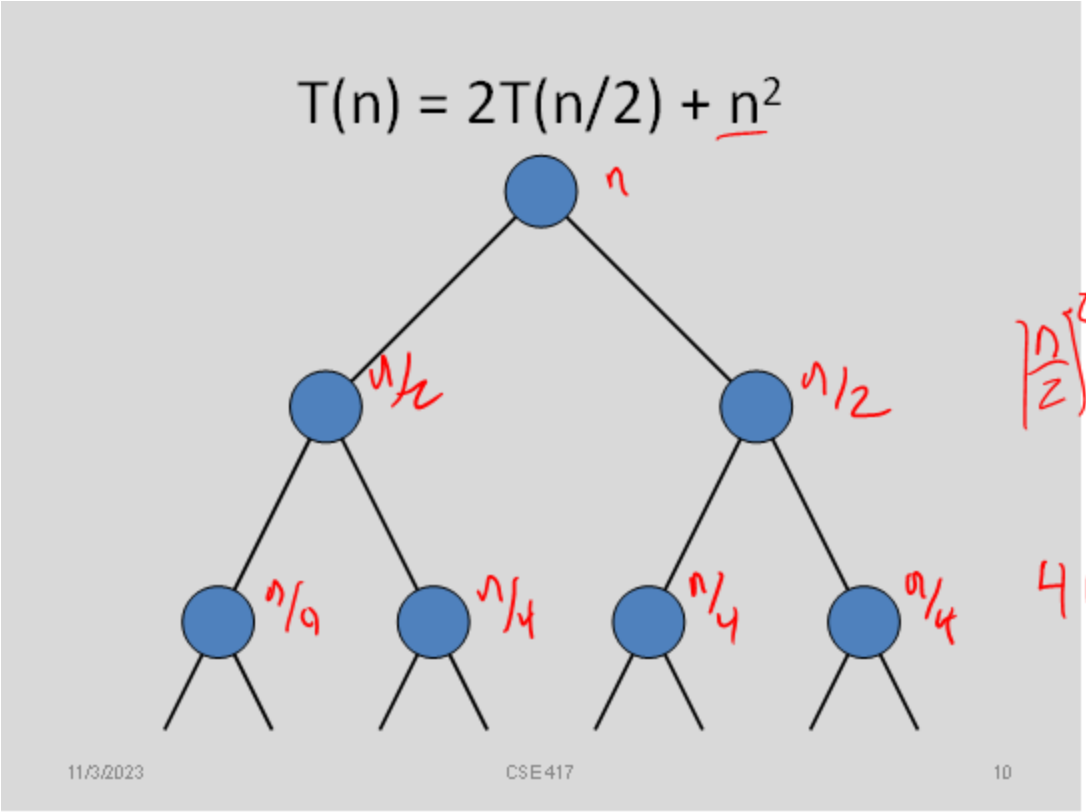
n

$\frac{n}{2} + \frac{n}{2} + \frac{n}{2} + \frac{n}{2} = 2n$

$16 \cdot \frac{n}{4} = 4n$

$8n$



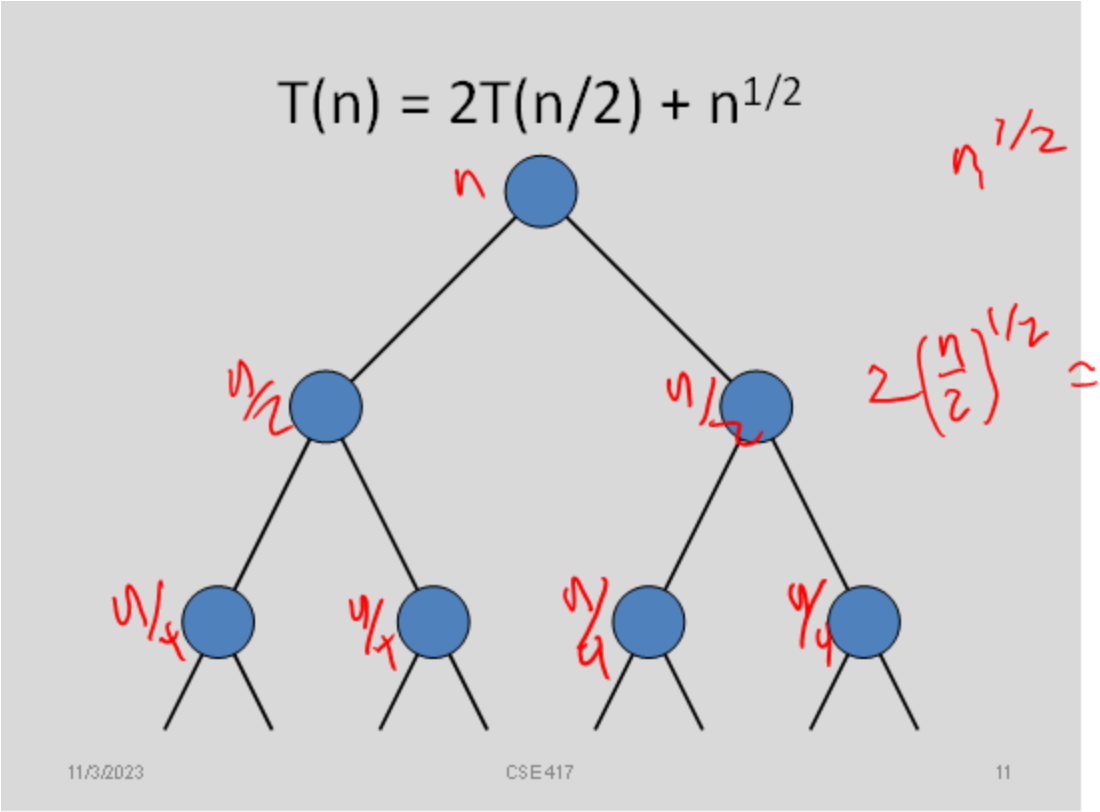
n^2

$\left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}\right)^2 \quad \frac{n^2}{2}$

$4\left(\frac{n}{4}\right)^2 = \frac{4n^2}{16} = \frac{n^2}{4}$

$\frac{n^2}{8}$

$n^2\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots \frac{c}{n}\right) = O(n^2)$



$n^{1/2}$

$2\left(\frac{n}{2}\right)^{1/2} =$

$\frac{2}{\sqrt{2}} n^{1/2}$

$= \sqrt{2}\, n^{1/2}$

$2\, n^{1/2}$

Recurrences

- Three basic behaviors
  - Dominated by initial case
  - Dominated by base case
  - All cases equal – we care about the depth

Geometric Sum

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

$$= \frac{1 - x^{n+1}}{1 - x}$$

$x > 1$ x^n

$x < 1$ $\frac{1}{1-x}$

What you really need to know about recurrences

- Work per level changes geometrically with the level
- Geometrically increasing ($x > 1$)
  - The bottom level wins
- Geometrically decreasing ($x < 1$)
  - The top level wins
- Balanced ($x = 1$)
  - Equal contribution

Classify the following recurrences (Increasing, Decreasing, Balanced)

- $T(n) = n + 5T(n/8)$
- $T(n) = n + 9T(n/8)$
- $T(n) = n^2 + 4T(n/2)$
- $T(n) = n^3 + 7T(n/2)$
- $T(n) = n^{1/2} + 3T(n/4)$

Recursive Matrix Multiplication

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = ae + bf$
$s = ag + bh$
$t = ce + df$
$u = cg + dh$

A N x N matrix can be viewed as a 2 x 2 matrix with entries that are (N/2) x (N/2) matrices.

The recursive matrix multiplication algorithm recursively multiplies the (N/2) x (N/2) matrices and combines them using the equations for multiplying 2 x 2 matrices

Recursive Matrix Multiplication

- How many recursive calls are made at each level?
- How much work in combining the results?
- What is the recurrence?

What is the run time for the recursive Matrix Multiplication Algorithm?

- Recurrence:

Strassen's Algorithm

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = p_1 + p_2 - p_4 + p_6$

$s = p_4 + p_5$

$t = p_6 + p_7$

$u = p_2 - p_3 + p_5 - p_7$

Where:

$p_1 = (b - d)(f + h)$

$p_2 = (a + d)(e + h)$

$p_3 = (a - c)(e + g)$

$p_4 = (a + b)h$

$p_5 = a(g - h)$

$p_6 = d(f - e)$

$p_7 = (c + d)e$

From AHU 1974

Recurrence for Strassen's Algorithms

- $T(n) = 7\,T(n/2) + cn^2$
- What is the runtime?

$\log_2 7 \approx 2.8073549221$