CSE 417
Algorithms and Complexity

Autumn 2023

Lecture 16

Divide and Conquer and Recurrences

Divide and Conquer

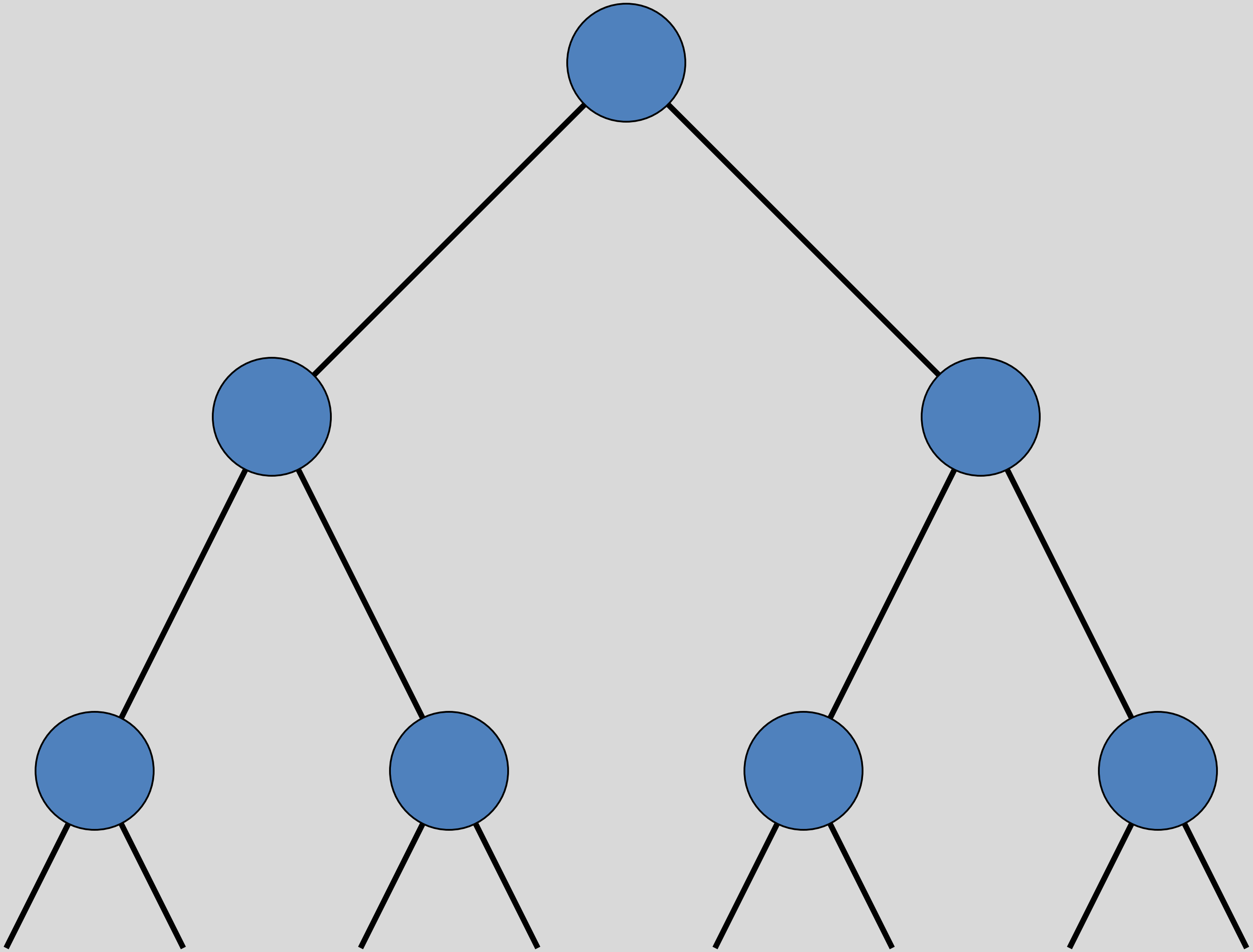
- Recurrences, Sections 5.1 and 5.2
- Algorithms
  - Median (Selection)
  - Fast Matrix Multiplication
  - Counting Inversions (5.3)
  - Multiplication (5.5)

Divide and Conquer : Merge Sort

```
Array MSort(Array a, int n){
    if (n <= 1)  return a;
    return Merge(MSort(a[0 .. n/2], n/2),  MSort(a[n/2+1 .. n-1], n/2);
}
```

$T(n) = 2T(n/2) + n; T(1) = 1;$

Unrolling the recurrence



$T(n) = 2T(n/2) + n; T(1) = 1;$

Substitution

Prove $T(n) \le n(\log_2 n + 1)$ for $n \ge 1$

Induction – Show P(1) and $\forall_{k<n} P(k) \implies P(n)$

Base Case: $T(1) = 1 = 1(\log_2 1 + 1)$

Induction: Assume $T(n/2) \le n/2(\log_2(n/2) + 1)$

$$
\begin{aligned}
T(n) &= 2\,T(n/2) + n \\
&\le 2\,n/2\,(\log_2(n/2) + 1) + n \\
&= n(\log_2 n - 1 + 1) + n \\
&= n(\log_2 n + 1)
\end{aligned}
$$

$T(n) = aT(n/b) + n^c$

Master Theorem

If $T(n) = aT(n/b) + O(n^d)$ for constants $a > 0$, $b > 1$, $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

$T(n) = T(n/2) + cn$

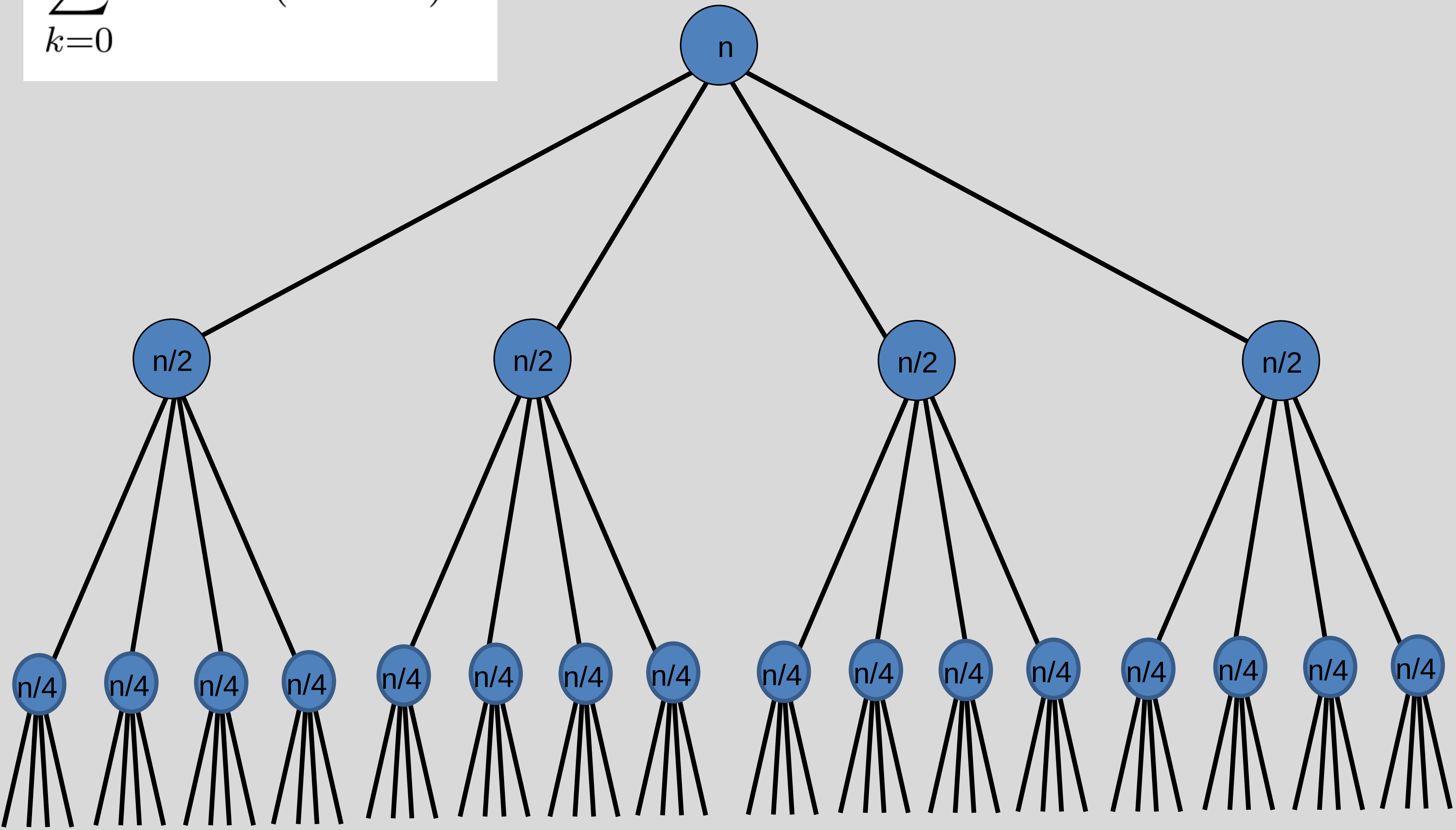
Where does this recurrence arise?

Solving the recurrence exactly

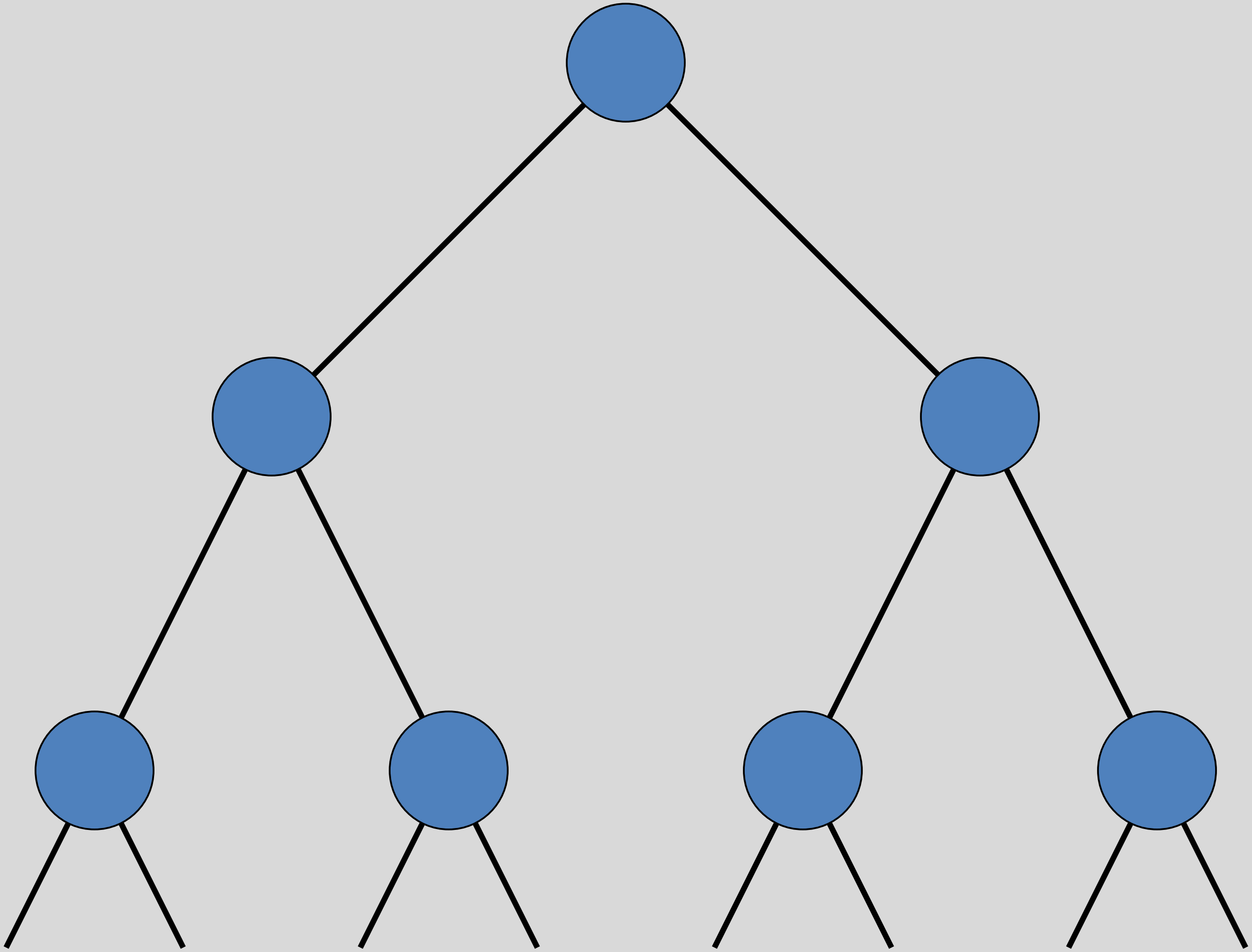
Total Work

$$\sum_{k=0}^{\log n} 2^k n = (2n-1)n$$

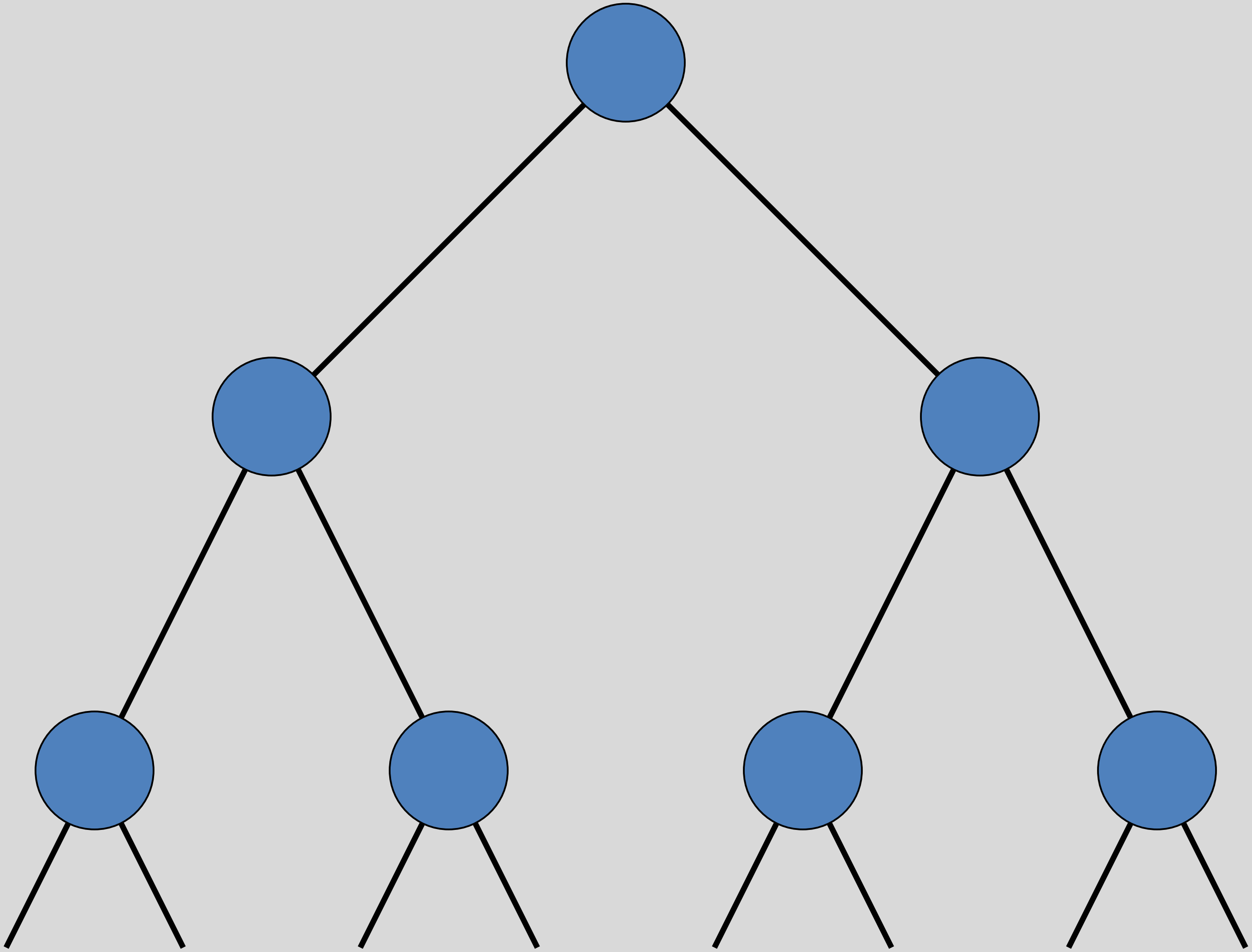
T(n) = 4T(n/2) + n



$T(n) = 2T(n/2) + n^2$



$$T(n) = 2T(n/2) + n^{1/2}$$



Recurrences

- Three basic behaviors
  - Dominated by initial case
  - Dominated by base case
  - All cases equal – we care about the depth

Geometric Sum

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

What you really need to know about recurrences

- Work per level changes geometrically with the level
- Geometrically increasing ($x > 1$)
  - The bottom level wins
- Geometrically decreasing ($x < 1$)
  - The top level wins
- Balanced ($x = 1$)
  - Equal contribution

Classify the following recurrences (Increasing, Decreasing, Balanced)

- $T(n) = n + 5T(n/8)$
- $T(n) = n + 9T(n/8)$
- $T(n) = n^2 + 4T(n/2)$
- $T(n) = n^3 + 7T(n/2)$
- $T(n) = n^{1/2} + 3T(n/4)$

Recursive Matrix Multiplication

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

r = ae + bf

s = ag + bh

t = ce + df

u = cg + dh

A N x N matrix can be viewed as a 2 x 2 matrix with entries that are (N/2) x (N/2) matrices.

The recursive matrix multiplication algorithm recursively multiplies the (N/2) x (N/2) matrices and combines them using the equations for multiplying 2 x 2 matrices

Recursive Matrix Multiplication

- How many recursive calls are made at each level?

- How much work in combining the results?

- What is the recurrence?

What is the run time for the recursive Matrix Multiplication Algorithm?

- Recurrence:

Strassen's Algorithm

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = p_1 + p_2 – p_4 + p_6$

$s = p_4 + p_5$

$t = p_6 + p_7$

$u = p_2 - p_3 + p_5 - p_7$

Where:

$p_1 = (b – d)(f + h)$

$p_2 = (a + d)(e + h)$

$p_3 = (a – c)(e + g)$

$p_4 = (a + b)h$

$p_5 = a(g – h)$

$p_6 = d(f – e)$

$p_7 = (c + d)e$

From AHU 1974

Recurrence for Strassen's Algorithms

- $T(n) = 7\,T(n/2) + cn^2$
- What is the runtime?

$\log_2 7 = 2.8073549221$