CSE 417
Algorithms and Complexity

Autumn 2023
Lecture 15
Divide and Conquer and Recurrences

Announcements

Divide and Conquer

- Recurrences, Sections 5.1 and 5.2
- Algorithms
  - Median (Selection)
  - Fast Matrix Multiplication
  - Counting Inversions (5.3)
  - Multiplication (5.5)

Divide and Conquer : Merge Sort

```
Array Mergesort(Array a){
        n = a.Length;
        if (n <= 1)
                return a;
        b = Mergesort(a[0 .. n/2]);
        c = Mergesort(a[n/2+1 .. n-1]);
        return Merge(b, c);
}
```

Algorithm Analysis

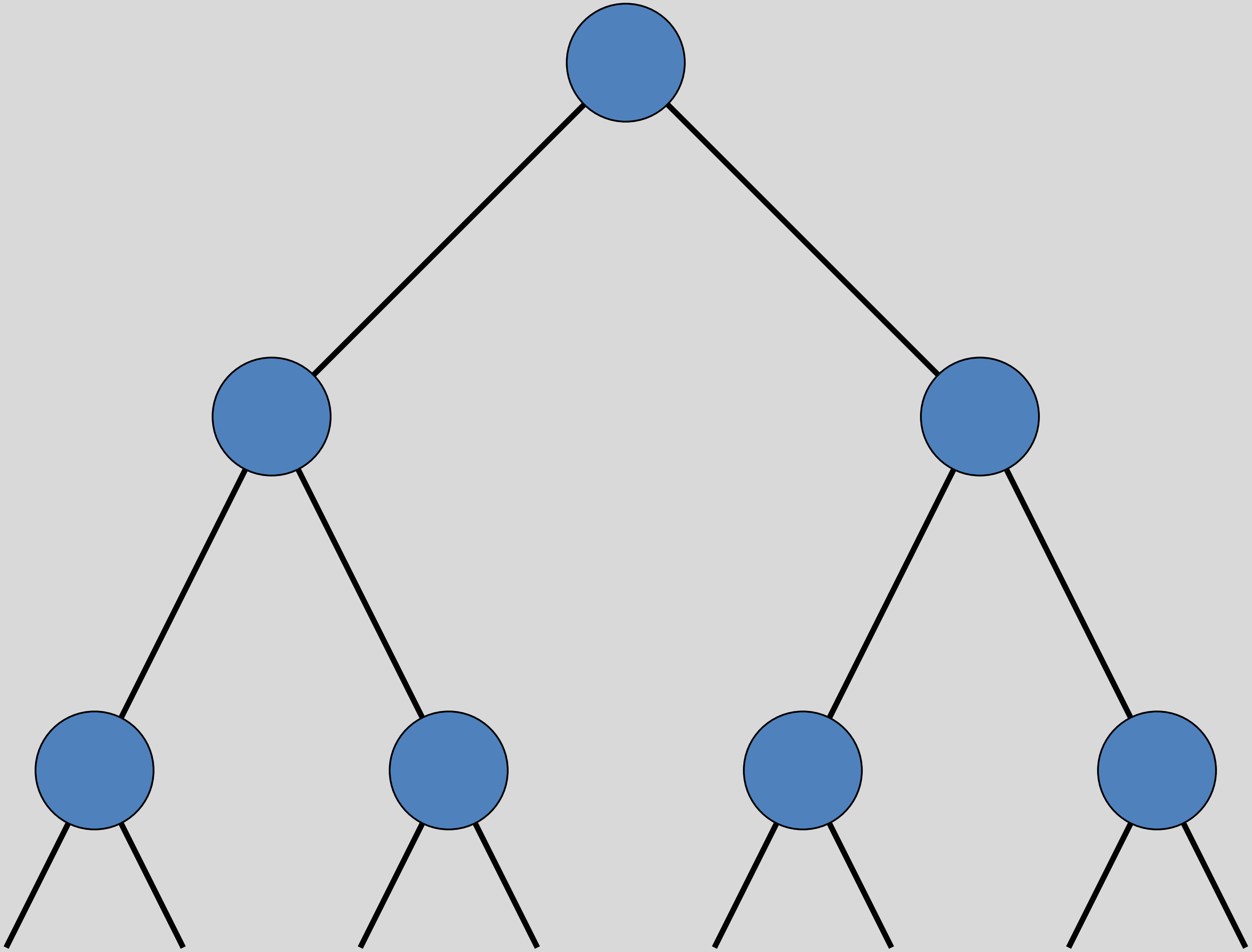
- Cost of Merge
- Cost of Mergesort

$$T(n) = 2T(n/2) + cn;\ T(1) = c;$$

Recurrence Analysis

- Solution methods
  - Unrolling recurrence
  - Guess and verify
  - Plugging in to a “Master Theorem”

Unrolling the recurrence



$T(n) = 2T(n/2) + n; T(1) = 1;$

Substitution

Prove $T(n) \leq n (\log_2 n + 1)$ for $n \geq 1$

Induction:

Base Case:

Induction Hypothesis:

A better mergesort (?)

- Divide into 3 subarrays and recursively sort
- Apply 3-way merge

What is the recurrence?

Unroll recurrence for $T(n) = 3T(n/3) + n$

$$T(n) = aT(n/b) + f(n)$$

$T(n) = T(n/2) + cn$

Where does this recurrence arise?

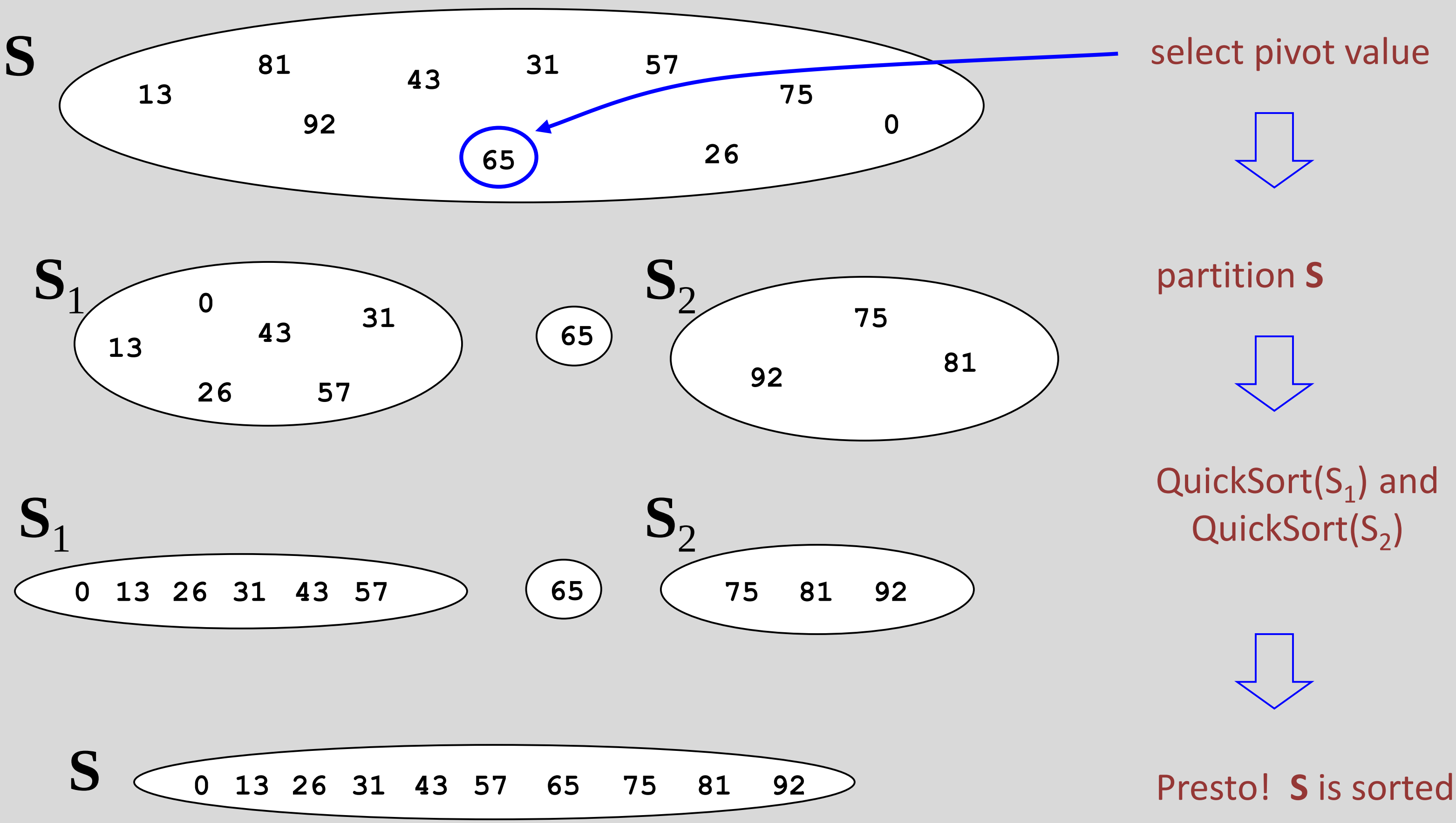
Quicksort

QuickSort(S):

1. Pick an element *v* in **S**. This is the ***pivot*** value.
2. Partition **S**-{*v*} into two disjoint subsets, $\mathbf{S}_1$ and $\mathbf{S}_2$ such that:
   - elements in $\mathbf{S}_1$ are all < *v*
   - elements in $\mathbf{S}_2$ are all > *v*
3. Return concatenation of QuickSort($\mathbf{S}_1$), *v*, QuickSort($\mathbf{S}_2$)

Recursion ends if Quicksort( ) receives an array of length 0 or 1.

The steps of Quicksort



Picking the pivot

- Choose the first element in the subarray
- Choose a value that might be close to the middle
  - Median of three
- Choose a random element

Recurrence for Quicksort

$$QS(n) = \sum_{i=1}^{n} \frac{1}{n} \{QS(i-1) + QS(n-i)\}$$

Computing the Median

- Given n numbers, find the number of rank n/2
- One approach is sorting
  - Sort the elements, and choose the middle one
  - Can you do better?

Problem generalization

- *Selection*, given n numbers and an integer k, find the k-th largest

Select(A, k)

Select(A, k){

  Choose element x from A

  $S_1 = \{y \text{ in } A \mid y < x\}$

  $S_2 = \{y \text{ in } A \mid y > x\}$

  $S_3 = \{y \text{ in } A \mid y = x\}$

  if ($|S_2| >= k$)

    return Select(S_2, k)

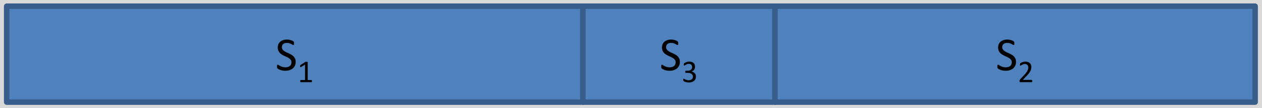
  else if ($|S_2| + |S_3| >= k$)

    return x

  else

    return Select(S_1, $k - |S_2| - |S_3|$)

}



Randomized Selection

- Choose the element at random
- Analysis can show that the algorithm has expected run time O(n)

$T(n) = T(n/2) + cn$

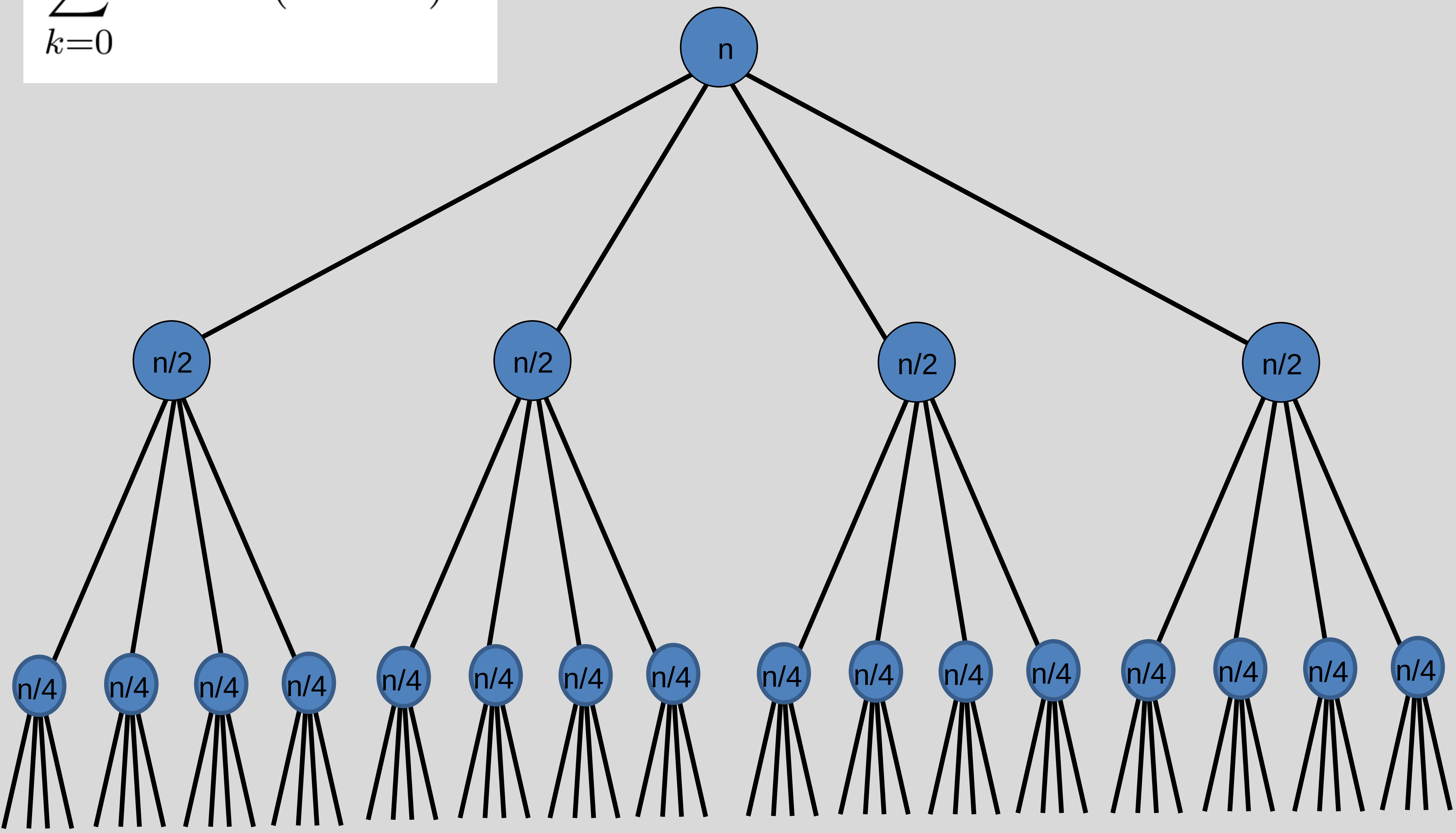
Where does this recurrence arise?

Solving the recurrence exactly

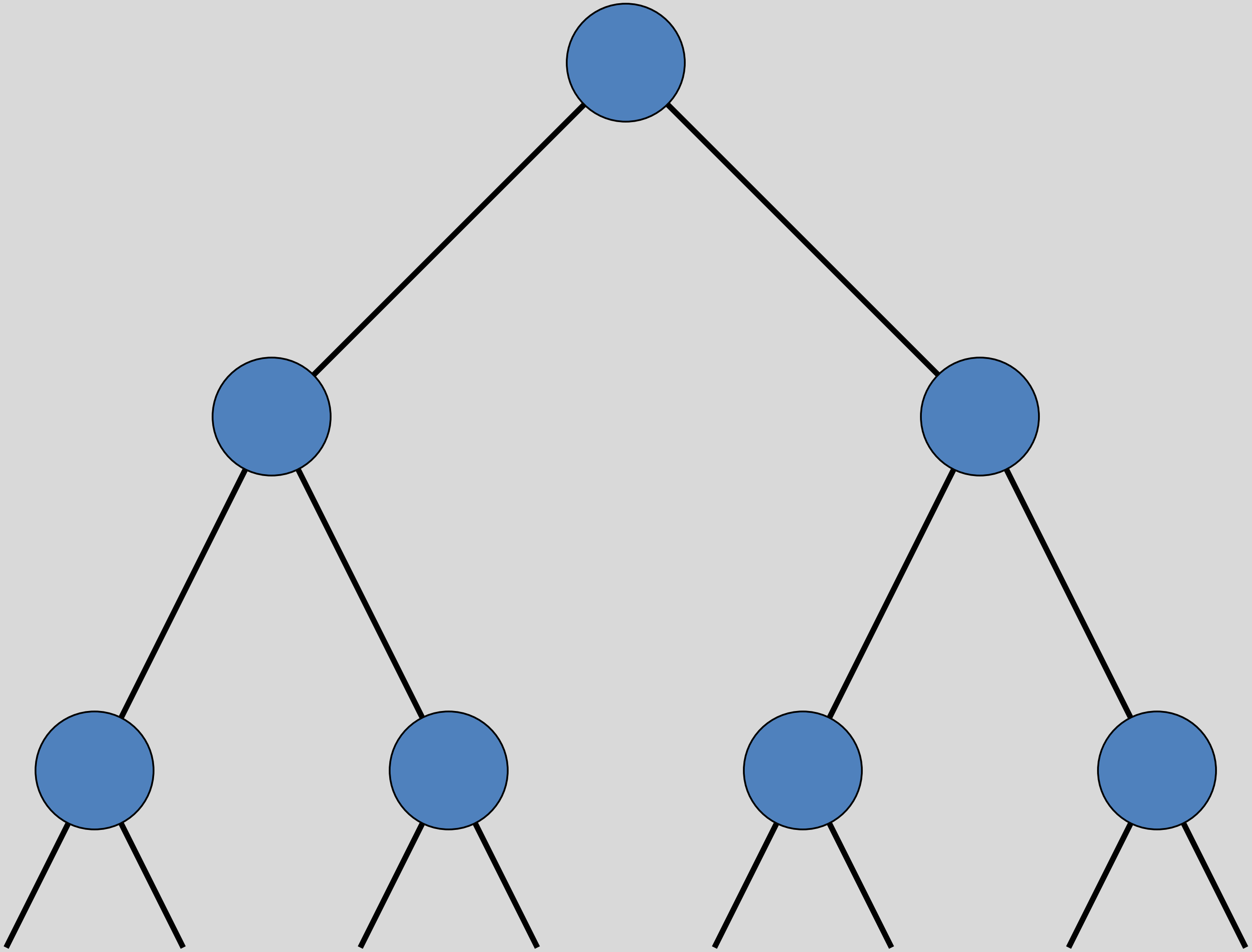
Total Work

$$\sum_{k=0}^{\log n} 2^k n = (2n - 1)n$$

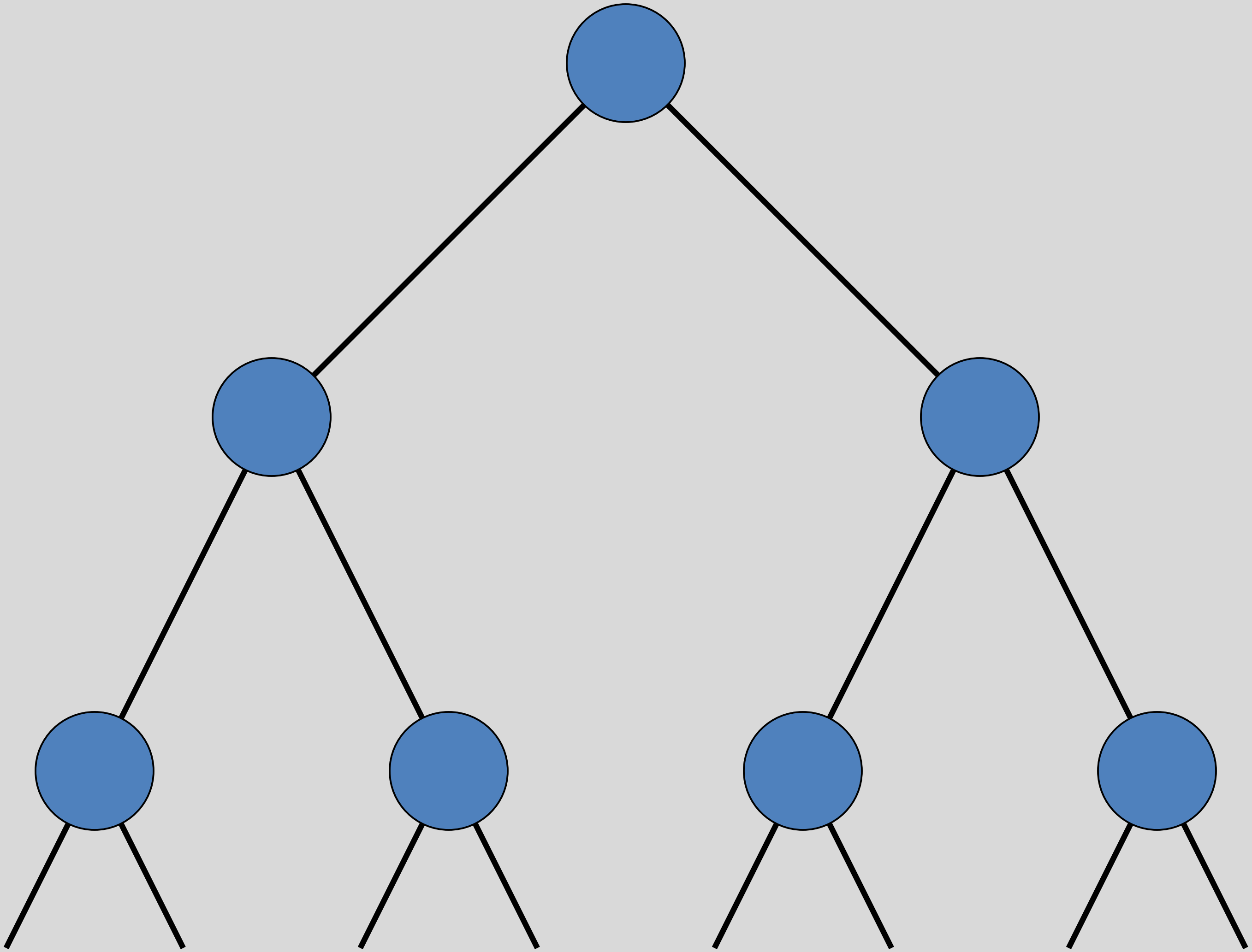
T(n) = 4T(n/2) + n



$T(n) = 2T(n/2) + n^2$



$$T(n) = 2T(n/2) + n^{1/2}$$



Recurrences

- Three basic behaviors
  - Dominated by initial case
  - Dominated by base case
  - All cases equal – we care about the depth

What you really need to know about recurrences

- Work per level changes geometrically with the level
- Geometrically increasing (x > 1)
  - The bottom level wins
- Geometrically decreasing (x < 1)
  - The top level wins
- Balanced (x = 1)
  - Equal contribution

Classify the following recurrences (Increasing, Decreasing, Balanced)

- $T(n) = n + 5T(n/8)$
- $T(n) = n + 9T(n/8)$
- $T(n) = n^2 + 4T(n/2)$
- $T(n) = n^3 + 7T(n/2)$
- $T(n) = n^{1/2} + 3T(n/4)$