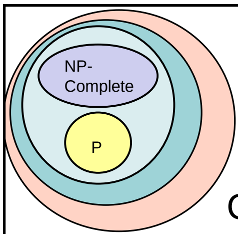
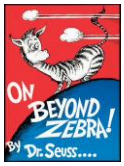



# CSE 417 Algorithms and Complexity

Richard Anderson - Lecture 27

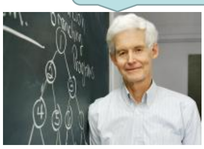
Coping with NP-Completeness and Beyond

# Announcements

- No final exam
- Homework 9
  - Due March 13, 5:00 pm
- Homework 10
  - Due March 18, 5:00 pm
  - NP-Completeness
  - Counts as a regular HW
- Office hours by zoom

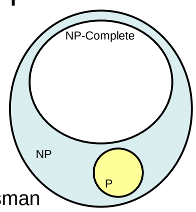
# NP Completeness: The story so far

Circuit Satisfiability is NP-Complete



There are a whole bunch of other important problems which are NP-Complete

# Populating the NP-Completeness Universe




- Circuit Sat $<_P$ 3-SAT
- 3-SAT $<_P$ Independent Set
- 3-SAT $<_P$ Vertex Cover
- Independent Set $<_P$ Clique
- 3-SAT $<_P$ Hamiltonian Circuit
- Hamiltonian Circuit $<_P$ Traveling Salesman
- 3-SAT $<_P$ Integer Linear Programming
- 3-SAT $<_P$ Graph Coloring
- 3-SAT $<_P$ Subset Sum
- Subset Sum $<_P$ Scheduling with Release times and deadlines

# Satisfiability

Literal: A Boolean variable or its negation. $x_i$ or $\overline{x_i}$

Clause: A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form: A propositional formula $\Phi$ that is the conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

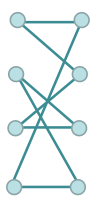
SAT: Given CNF formula $\Phi$, does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

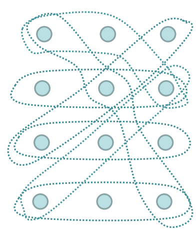
Ex: $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

Yes: $x_1$ = true, $x_2$ = true $x_3$ = false.

# Matching

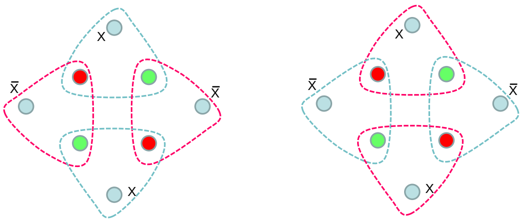


Two dimensional matching



Three dimensional matching (3DM)

## 3-SAT $<_P$ 3DM

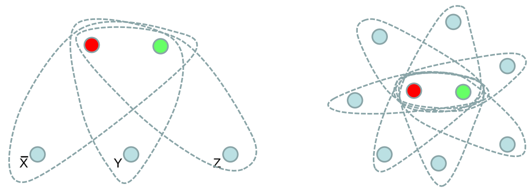



X True

X False

Truth Setting Gadget

## 3-SAT $<_P$ 3DM




Clause gadget for ($\overline{X}$ OR Y OR Z)

Garbage Collection Gadget
(Many copies)

## Exact Cover (sets of size 3) XC3

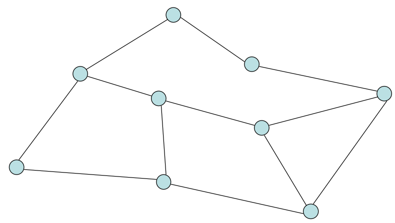
Given a collection of sets of size 3 of a domain of size 3N, is there a sub-collection of N sets that cover the sets

(A, B, C), (D, E, F), (A, B, G),
(A, C, I), (B, E, G), (A, G, I),
(B, D, F), (C, E, I), (C, D, H),
(D, G, I), (D, F, H), (E, H, I),
(F, G, H), (F, H, I)

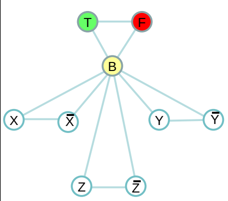
3DM $<_P$ XC3

## Graph Coloring

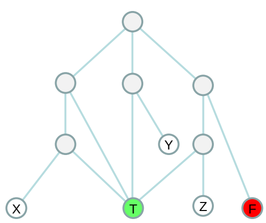
- NP-Complete
  - Graph K-coloring
  - Graph 3-coloring
- Polynomial
  - Graph 2-Coloring



## 3-SAT $<_P$ 3 Colorability




Truth Setting Gadget




Clause Testing Gadget

(Can be colored if at least one input is T)

## Number Problems

- Subset sum problem
  - Given natural numbers $w_1, \ldots, w_n$ and a target number W, is there a subset that adds up to exactly W?
- Subset sum problem is NP-Complete
- Subset Sum problem can be solved in O(nW) time

# XC3 $<_P$ SUBSET SUM

Idea: Represent each set as a large integer, where the element $x_i$ is encoded as $D^i$ where D is an integer

$\{x_3, x_5, x_9\} \Rightarrow D^3 + D^5 + D^9$

Does there exist a subset that sums to exactly
$D^1 + D^2 + D^3 + \ldots + D^{n-1} + D^n$

Detail: How large is D? We need to make sure that we do not have any carries, so we can choose D = m+1, where m is the number of sets.

# Integer Linear Programming

- Linear Programming – maximize a linear function subject to linear constraints
- Integer Linear Programming – require an integer solution
- NP Completeness reduction from 3-SAT

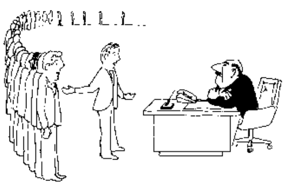
Use 0-1 variables for $x_i$'s

Constraint for clause $x_1 \vee \overline{x_2} \vee \overline{x_3}$

$x_1 + (1 - x_2) + (1 - x_3) > 0$

# Coping with NP-Completeness

- Approximation Algorithms
- Exact solution via Branch and Bound
- Local Search

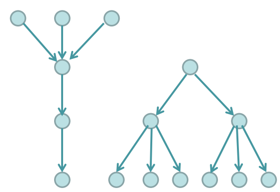


I can't find an efficient algorithm, but neither can all these famous people.

# Multiprocessor Scheduling

- Unit execution tasks
- Precedence graph
- K-Processors

- Polynomial time for k=2
- Open for k = constant
- NP-complete is k is part of the problem



# Highest level first is 2-Optimal

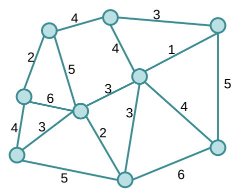
Choose k items on the highest level

Claim: number of rounds is at least twice the optimal.

# Christofides TSP Algorithm

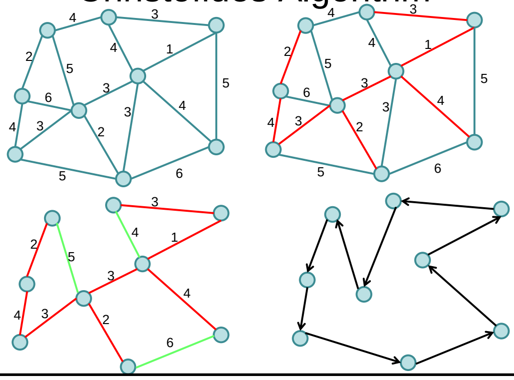
- Undirected graph satisfying triangle inequality

1. Find MST
2. Add additional edges so that all vertices have even degree
3. Build Eulerian Tour




3/2 Approximation

# Christofides Algorithm



# Bin Packing

- Given N items with weight $w_i$, pack the items into as few unit capacity bins as possible
- Example: .3, .3, .3, .3, .4, .4

# First Fit Packing

- First Fit
  - Theorem: FF(I) is at most 17/10 Opt(I) + 2
- First Fit Decreasing
  - Theorem: FFD(I) is at most 11/9 Opt (I) + 4

# Branch and Bound

- Brute force search – tree of all possible solutions
- Branch and bound – compute a lower bound on all possible extensions
  - Prune sub-trees that cannot be better than optimal

# Branch and Bound for TSP

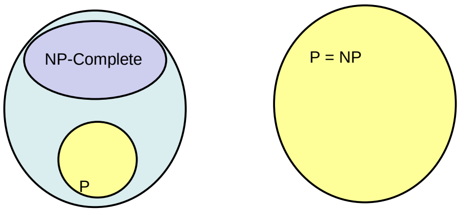
- Enumerate all possible paths
- Lower bound, Current path cost plus MST of remaining points
- Euclidean TSP
  - Points on the plane with Euclidean Distance
  - Sample data set: State Capitals

# Local Optimization

- Improve an optimization problem by local improvement
  - Neighborhood structure on solutions
  - Travelling Salesman 2-Opt (or K-Opt)
  - Independent Set Local Replacement

## What we don't know

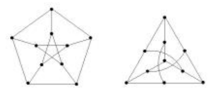
- P vs. NP




## If P != NP, is there anything in between

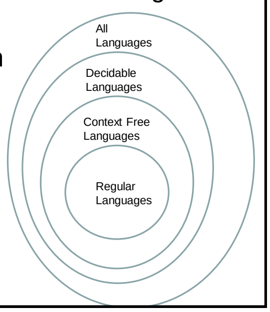
- Yes, Ladner [1975]
- Problems not known to be in P or NP Complete
  - Factorization
  - Discrete Log
  - Graph Isomorphism

Solve $g^k = b$ over a finite group



## Complexity Theory

- Computational requirements to recognize languages
- Models of Computation
- Resources
- Hierarchies



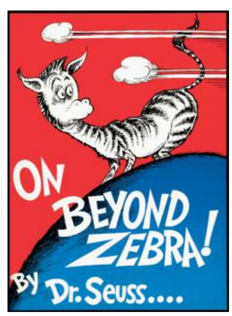

## Time complexity

- P: (Deterministic) Polynomial Time
- NP: Non-deterministic Polynomial Time
- EXP: Exponential Time

## Space Complexity

- Amount of Space (Exclusive of Input)
- L: Logspace, problems that can be solved in O(log n) space for input of size n
  - Related to Parallel Complexity
- PSPACE, problems that can be required in a polynomial amount of space

## So what is beyond NP?



## NP vs. Co-NP

- Given a Boolean formula, is it true for some choice of inputs
- Given a Boolean formula, is it true for all choices of inputs

## Problems beyond NP

- Exact TSP, Given a graph with edge lengths and an integer K, does the minimum tour have length K
- Minimum circuit, Given a circuit C, is it true that there is no smaller circuit that computes the same function a C

## Polynomial Hierarchy

- Level 1
  - $\exists X_1\ \Phi(X_1)$, $\forall X_1\ \Phi(X_1)$
- Level 2
  - $\forall X_1 \exists X_2\ \Phi(X_1,X_2)$, $\exists X_1 \forall X_2\ \Phi(X_1,X_2)$
- Level 3
  - $\forall X_1 \exists X_2 \forall X_3\ \Phi(X_1,X_2,X_3)$, $\exists X_1 \forall X_2 \exists X_3\ \Phi(X_1,X_2,X_3)$

## Polynomial Space

- Quantified Boolean Expressions
  - $\exists X_1 \forall X_2 \exists X_3 \ldots \exists X_{n-1} \forall X_n\ \Phi(X_1,X_2,X_3 \ldots X_{n-1}X_n)$
- Space bounded games
  - Competitive Facility Location Problem
  - N x N Chess
- Counting problems
  - The number of Hamiltonian Circuits